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On metric spaces where continuous real valued functions are uniformly continuous in **ZF**

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A R T I C L E I N F O

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We show that the negation of each one of the following statements is consistent with **ZF**:

(i) Every sequentially compact metric space $\mathbf{X} = (X, d)$ is normal, i.e., the distance of any two disjoint non-empty closed subsets of $\mathbf{X}$ is strictly positive.

(ii) If (X, d) is a sequentially compact metric space then $\mathbf{X}$ is a UC space, i.e., every continuous real valued on $\mathbf{X}$ is uniformly continuous.

(iii) If (X, d) is a UC metric space then $\mathbf{X}$ is Lebesgue, i.e., every open cover of $\mathbf{X}$ has a Lebesgue number.

(iv) If (X, d) is a metric space such that every countable open cover of $\mathbf{X}$ has a Lebesgue number then $\mathbf{X}$ is Lebesgue.

We also show:

(v) For every metric space $\mathbf{X}$, the following are equivalent:
(1) Every sequence in $\mathbf{X}$ admits a Cauchy subsequence;
(2) For every sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{X}$, for each $\varepsilon > 0$ there is an infinite $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \in \mathbb{N}_\varepsilon$;
(3) For every sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{X}$, for each $\varepsilon > 0$ and for each $n_0 \in \mathbb{N}$ there exist $n, m \in \mathbb{N}$, $n, m \geq n_0$, $n \neq m$ such that $d(x_n, x_m) < \varepsilon$.

(vi) The axiom of countable choice **CAC** implies that for every metric space $\mathbf{X}$ the following statements are equivalent:
(1) $\mathbf{X}$ is Lebesgue;
(2) Every countable open cover of $\mathbf{X}$ has a Lebesgue number;
(3) $\mathbf{X}$ is UC.

1. Notation and terminology

Let $\mathbf{X} = (X, d)$ be a metric space, $x \in X$, $\varepsilon > 0$ and $B \subseteq X$. $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ denotes the *open ball* in $\mathbf{X}$ with center x and radius ε and, $\delta(B) = \sup\{d(x, y) : x, y \in B\} \in \mathbb{R}_+ \cup \{+\infty\}$ denotes the

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diameter of B. We say that an open cover $\mathcal{U}$ of $\mathbf{X}$ has a *Lebesgue number* $\delta > 0$ iff for every $A \subseteq X$ with $\delta(A) < \delta$ there exists $U \in \mathcal{U}$ with $A \subseteq U$.

$\mathbf{X}$ is said to be *bounded* iff the diameter $\delta(X) < +\infty$.

$\mathbf{X}$ is said to be *Lebesgue* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a Lebesgue number.

$\mathbf{X}$ is said to be *countably Lebesgue* iff every countable open cover $\mathcal{U}$ of $\mathbf{X}$ has a Lebesgue number.

$\mathbf{X}$ is said to be *UC* or *Atsuji* iff every continuous real valued function on $\mathbf{X}$ is uniformly continuous.

$\mathbf{X}$ is said to be *normal* iff the distance of every two disjoint, non-empty closed subsets of $\mathbf{X}$ is strictly positive.

$\mathbf{X}$ is said to be *selective* iff the family of all non-empty open sets has a choice set. Equivalently, see [6], $\mathbf{X}$ is selective iff $\mathbf{X}$ has a well ordered dense subset.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{X}$. $(x_n)_{n\in\mathbb{N}}$ is said to be *cofinally Cauchy* if for each $\varepsilon > 0$ there is an infinite $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \in \mathbb{N}_\varepsilon$. Equivalently, $(x_n)_{n\in\mathbb{N}}$ is cofinally Cauchy iff for every $\varepsilon > 0$ there is some open ball $B(x, \varepsilon)$ in $\mathbf{X}$ such that there are infinitely many $n \in \mathbb{N}$ such that $x_n \in B(x, \varepsilon)$. $(x_n)_{n\in\mathbb{N}}$ is said to be *pseudo-Cauchy* if for each $\varepsilon > 0$ and for each $n_0 \in \mathbb{N}$ there exist $n, m \in \mathbb{N}$, $n, m \geq n_0$, $n \neq m$ such that $d(x_n, x_m) < \varepsilon$.

$\mathbf{X}$ has the *cofinally Cauchy property* (resp. the *pseudo-Cauchy property*) iff each of its sequences is cofinally Cauchy (resp. each of its sequences is pseudo-Cauchy). If $\mathbf{X}$ has the cofinally Cauchy property (resp. the pseudo-Cauchy property) then we say that $\mathbf{X}$ is *cofinally Cauchy* (resp. *pseudo-Cauchy*). It is easy to see that each cofinally Cauchy sequence in $\mathbf{X}$ is pseudo-Cauchy. So, if $\mathbf{X}$ is cofinally Cauchy then $\mathbf{X}$ is pseudo-Cauchy. Note that there are metric spaces having pseudo-Cauchy sequences which fail to be cofinally Cauchy. Indeed, the sequence of real numbers $(a_n)_{n\in\mathbb{N}}$, where for every $n \in \mathbb{N}$,

$$a_n = \begin{cases} k + 1/k, \text{ if } n = 2k - 1 \text{ for some } k \in \mathbb{N} \\ k + 1/(k+1), \text{ if } n = 2k \text{ for some } k \in \mathbb{N} \end{cases}$$

is such a sequence. The notions of cofinally Cauchy and pseudo-Cauchy sequences are generalizations of the notion of Cauchy sequence and are due to G. Beer, [3] and Toader, [14] respectively.

$\mathbf{X}$ is said to be *Heine–Borel compact* or simply *compact* if every open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

$\mathbf{X}$ is said to be *countably compact* if every countable open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$. Equivalently, $\mathbf{X}$ is countably compact iff for every countable family of closed subsets of $\mathbf{X}$ having the *finite intersection property* (fip for abbreviation) has a non-empty intersection.

$\mathbf{X}$ is *complete* or *Frechét complete* iff every Cauchy sequence of points of X converges to some element of X.

$\mathbf{X}$ is *sequentially compact* iff every sequence has a convergent subsequence.

$\mathbf{X}$ is *totally bounded* iff for every real number $\varepsilon > 0$, there exists an ε-net, i.e., a finite subset $\{x_i : i \leq n\}$ of X such that $\bigcup\{B(x_i, \varepsilon) : i \leq n\} = X$. Clearly, each totally bounded metric space is bounded, but the converse is not true in general. For example, an infinite set equipped with the discrete metric is bounded but not totally bounded.

$\mathbf{X}$ is *sequentially bounded* or *Cauchy-precompact* iff every sequence admits a Cauchy subsequence.

$\mathbf{X}$ is *cofinally complete* if every cofinally Cauchy sequence in $\mathbf{X}$ has a cluster point.

Let $\mathcal{C}$, $\mathcal{SC}$, $\mathcal{TB}$, $\mathcal{SB}$, $\mathcal{CTB}$, $\mathcal{CSB}$, $\mathcal{CC}$, $\mathcal{LTB}$, $\mathcal{UC}$, $\mathcal{L}$, $\mathcal{CL}$, $\mathcal{N}$, *pseud*$\mathcal{C}$ and *cof*$\mathcal{C}$ denote the classes of all Heine–Borel compact, sequentially compact, totally bounded, sequentially bounded, complete and totally bounded, complete and sequentially bounded, countably compact, Lebesgue and totally bounded, *UC*, Lebesgue, countably Lebesgue, normal, pseudo-Cauchy and cofinally Cauchy metric spaces respectively.

An infinite set X is said to be *Dedekind-infinite*, denoted by $\mathbf{DI}(X)$, iff X contains a countably infinite set. Otherwise is said to be *Dedekind-finite*. By universal quantifying over X, $\mathbf{DI}(X)$ gives rise to the choice principle $\mathbf{IDI} : \forall X(X \text{ infinite } \to \mathbf{DI}(X))$ that is, "every infinite set is Dedekind-infinite" (Form 9 of [5]).

Below we list some of weak forms of the axiom of choice we shall deal with in the sequel.