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A Garden of Eden theorem for Anosov diffeomorphisms on tori

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ABSTRACT

Let f be an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$ and τ a continuous self-mapping of $\mathbb{T}^n$ commuting with f. We prove that τ is surjective if and only if the restriction of τ to each homoclinicity class of f is injective.

1. Introduction

The Garden of Eden theorem, originally established by Moore [21] and Myhill [22] in the early 1960s, is an important result in symbolic dynamics and coding theory. It provides a necessary and sufficient condition for a cellular automaton to be surjective. More specifically, consider a finite set A and the set $A^{\mathbb{Z}}$ consisting of all bi-infinite sequences $x = (x_i)$ with $x_i \in A$ for all $i \in \mathbb{Z}$. We equip $A^{\mathbb{Z}}$ with its *prodiscrete topology*,

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that is, with the topology of pointwise convergence (this is also the product topology obtained by taking the discrete topology on each factor A of $A^{\mathbb{Z}}$). A *cellular automaton* is a continuous map $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ that commutes with the shift homeomorphism $\sigma\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $\sigma(x) = (x_{i-1})$ for all $x = (x_i) \in A^{\mathbb{Z}}$. Two sequences $x = (x_i), y = (y_i) \in A^{\mathbb{Z}}$ are said to be *almost equal* if one has $x_i = y_i$ for all but finitely many $i \in \mathbb{Z}$. A cellular automaton $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is called *pre-injective* if there exist no distinct sequences $x, y \in A^{\mathbb{Z}}$ that are almost equal and satisfy $\tau(x) = \tau(y)$. The Moore–Myhill Garden of Eden theorem states that a cellular automaton $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is surjective if and only if it is pre-injective. The implication surjective $\Rightarrow$ pre-injective was first established by Moore [21], and Myhill [22] proved the converse implication shortly after.

The Moore–Myhill Garden of Eden theorem has been extended in several directions. There are now versions of it for cellular automata over amenable groups [19,9], cellular automata over subshifts [15,11,12], and linear cellular automata over linear shifts and subshifts [6,8] (the reader is referred to the monograph [7] for a detailed exposition of some of these extensions, as well as historical comments and additional references).

In this note, we present an analogue of the Garden of Eden theorem for Anosov diffeomorphisms on tori. This reveals one more connection between symbolic dynamics and the theory of smooth dynamical systems. Actually our motivation came from a phrase of Gromov [15, p. 195] which mentioned the possibility of extending the Garden of Eden theorem to a suitable class of hyperbolic dynamical systems.

Let (X, f) be a dynamical system consisting of a compact metrizable space X equipped with a homeomorphism $f\colon X \to X$. Two points in X are called f-*homoclinic* if their f-orbits are asymptotic both in the past and the future (see Section 2 for a precise definition). Homoclinicity defines an equivalence relation on X. An *endomorphism* of the dynamical system (X, f) is a continuous map $\tau\colon X \to X$ commuting with f. We say that an endomorphism τ of (X, f) is *pre-injective* (with respect to f) if the restriction of τ to each f-homoclinicity class is injective (i.e., there is no pair of distinct f-homoclinic points in X having the same image under τ) (in the particular case when $X = A^{\mathbb{Z}}$ and $f = \sigma$ is the shift homeomorphism, the endomorphisms of (X, f) are precisely the cellular automata and this definition of pre-injectivity is equivalent to the one given above, see e.g. [5, Proposition 2.5]). We say that the dynamical system (X, f) has the *Moore property* if every surjective endomorphism of (X, f) is pre-injective and that (X, f) has the *Myhill property* if every pre-injective endomorphism of (X, f) is surjective. We say that the dynamical system (X, f) has the *Moore–Myhill property*, or that it satisfies the *Garden of Eden theorem*, if (X, f) has both the Moore and the Myhill properties.

A C^1-diffeomorphism f of a compact C^r-differentiable ($r \geq 1$) manifold M is called an *Anosov diffeomorphism* if the tangent bundle of M splits as a direct sum $TM = E_s \oplus E_u$ of two invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M, the differential df is uniformly contracting on E_s and uniformly expanding on E_u (see [24,4,10,16,23]).

Our main result is the following.

Theorem 1.1 *(Garden of Eden theorem for toral Anosov diffeomorphisms). Let f be an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$. Then the dynamical system $(\mathbb{T}^n, f)$ has the Moore–Myhill property. In other words, if $\tau\colon \mathbb{T}^n \to \mathbb{T}^n$ is a continuous map commuting with f, then τ is surjective if and only if the restriction of τ to each homoclinicity class of f is injective.*

The paper is organized as follows. In Section 2, we fix notation and present some background material on dynamical systems. In Section 3, we establish Theorem 1.1. The proof uses two classical results in the theory of hyperbolic dynamical systems. The first one is the Franks–Manning theorem [13,20], which states that any Anosov diffeomorphism on $\mathbb{T}^n$ is topologically conjugate to a hyperbolic toral automorphism. The second is a theorem due to Walters [25] which asserts that all endomorphisms of a hyperbolic toral endomorphism are affine. This allows us to reduce the proof to an elementary question in linear algebra. In