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Two Cardinal inequalities about bidiscrete systems

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ABSTRACT

We consider the cardinal invariant bd defined by M. Džamonja and I. Juhász concerning bidiscrete systems. Using the relation between bidiscrete systems and irredundance for a compact Hausdorff space K, we prove that $w(K) \leq bd(K) \cdot hL(K)^+$, generalizing a result of S. Todorcevic concerning the irredundance in Boolean algebras and we prove that for every maximal irredundant family $\mathcal{F} \subset C(K)$, there is a π-base $\mathcal{B}$ for K with $|\mathcal{F}| = |\mathcal{B}|$, a result analogous to the McKenzie Theorem for Boolean algebras in the context of compact spaces. In particular, it is a consequence of the latter result that $\pi(K) \leq bd(K)$ for every compact Hausdorff space K. From the relation between bidiscrete systems and biorthogonal systems, we obtain some results about biorthogonal systems in Banach spaces of the form $C(K)$.

Introduction

In this work, we consider the notion of bidiscrete systems defined by M. Džamonja and I. Juhász in [2]:

Definition 1 *(M. Džamonja, I. Juhász [2]).* Let K be a compact Hausdorff space. A sequence $S = \{(x^0_\alpha, x^1_\alpha) : \alpha < \kappa\}$ of pairs of points in K (i.e., a subfamily of K^2) is called a bidiscrete system in K if there exists a family $\{f_\alpha : \alpha < \kappa\}$ of real valued continuous functions on K satisfying for every $\alpha, \beta < \kappa$:

- $f_\alpha(x^l_\alpha) = l$ for $l \in \{0, 1\}$,
- if $\alpha \neq \beta$ then $f_\alpha(x^0_\beta) = f_\alpha(x^1_\beta)$.

The cardinal invariant $bd(K)$ is defined to be

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$$bd(K) := \sup\{|S| : S \text{ is a bidiscrete system in } K\}.$$

The purpose of this work is to obtain some new cardinal inequalities for a compact Hausdorff space K, relating the cardinal invariant $bd(K)$ and some other topological cardinal invariants of K. Firstly, we will translate the definition of $bd(K)$ in terms of Banach spaces and in terms of Banach algebras. Given a compact Hausdorff space K, we consider the Banach space $C(K)$ of all real valued continuous functions on K with the supremum norm. In [2], the authors considered the notion of a nice biorthogonal system for Banach spaces of the form $C(K)$. We say that a sequence $(f_\alpha, \mu_\alpha)_{\alpha<\kappa}$ in $C(K) \times M(K)$ (where $M(K)$ is the space of Radon measures on K) is a biorthogonal system if for every $\alpha, \beta < \kappa$

$$\mu_\alpha(f_\beta) = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Moreover, if for each $\alpha < \kappa$, there are distinct points $x_\alpha, y_\alpha \in K$ such that $\mu_\alpha = \delta_{x_\alpha} - \delta_{y_\alpha}$, where δ_x denotes a Dirac measure centred in x, we say that the sequence $(f_\alpha, \mu_\alpha)_{\alpha<\kappa}$ is a nice biorthogonal system.

We observe that, if $\{(x_\alpha, y_\alpha) : \alpha < \kappa\}$ is a bidiscrete system in K, then by definition, there exists a family of functions $\{f_\alpha : \alpha < \kappa\}$ such that $(f_\alpha, \delta_{x_\alpha} - \delta_{y_\alpha})_{\alpha<\kappa}$ is a nice biorthogonal system. In the same way, if $(f_\alpha, \delta_{x_\alpha} - \delta_{y_\alpha})_{\alpha<\kappa}$ is a nice biorthogonal system, then it is easy to see that $\{(x_\alpha, y_\alpha) : \alpha < \kappa\}$ is a bidiscrete system. From this observation we conclude that

$$bd(K) = \sup\{\kappa : \text{ there is a nice biorthogonal system of size } \kappa \text{ in } C(K)\}.$$

In other words, the cardinal $bd(K)$ is equal to the cardinal $nbiort_2(K)$ as defined in [5].

For a compact Hausdorff space K, the corresponding Banach space $C(K)$ is a Banach algebra, where the multiplication is the pointwise multiplication of functions. A set $\mathcal{F} \subset C(K)$ is said to be irredundant if and only if for every $f \in \mathcal{F}$, there exists a Banach subalgebra $\mathcal{B} \subset C(K)$ such that $\mathcal{F} \setminus \{f\} \subset \mathcal{B}$ and $f \notin \mathcal{B}$. This is equivalent to say that, for every $f \in \mathcal{F}$, f does not belong to the Banach subalgebra generated by $\mathcal{F} \setminus \{f\}$, where the Banach subalgebra generated by $\mathcal{F} \setminus \{f\}$ is the smallest Banach subalgebra of $C(K)$ containing $\mathcal{F} \setminus \{f\}$. From Theorem 5.4 of [5], for a compact Hausdorff space K, $C(K)$ contains an irredundant set of size κ if and only if it contains a nice biorthogonal system of size κ. This implies that

$$bd(K) = \sup\{|\mathcal{F}| : \mathcal{F} \subset C(K) \text{ is an irredundant set of } C(K)\}.$$

We conclude that for every compact Hausdorff space K, there is a bidiscrete system in K of size κ if and only if there is an irredundant set of size κ in $C(K)$. This allows us to work in the frame of irredundant sets instead of bidiscrete systems. The definition of $bd(K)$ in terms of irredundant sets is similar to the definition of the well know cardinal invariant $irr(\mathcal{A})$ for a Boolean algebra $\mathcal{A}$. The cardinal invariant $irr(\mathcal{A})$ is defined in analogy to bd, where a subset $B \subset \mathcal{A}$ is irredundant if and only if for every $b \in B$, b does not belong to the Boolean subalgebra generated by $B \setminus \{b\}$. See [8] for definitions and some properties of $irr(\mathcal{A})$ for a Boolean algebra $\mathcal{A}$. In particular, if $\mathcal{A}$ is a Boolean algebra and if $K_\mathcal{A}$ denotes its Stone space, we have that $irr(\mathcal{A}) \leq bd(K_\mathcal{A})$. In fact, suppose $B \subset \mathcal{A}$ is an irredundant set. Then $\mathcal{F} := \{\chi_{[b]} : b \in B\}$ is an irredundant set in $C(K_\mathcal{A})$, where $[b]$ denotes the clopen set of $K_\mathcal{A}$ determined by $b \in B$. However, we do not know if we have the equality $irr(\mathcal{A}) = bd(K_\mathcal{A})$. Theorem 5.4 of [5] tells us that for every Boolean algebra $\mathcal{A}$, $bd(K_\mathcal{A}) \leq s(K^2_\mathcal{A})$, where $s(K_\mathcal{A})$ is the spread of $K_\mathcal{A}$. This implies that if $\mathcal{A}$ is a counterexample for the equality $irr(\mathcal{A}) = bd(K_\mathcal{A})$, then $irr(\mathcal{A}) < s(K^2_\mathcal{A})$. For example, the Boolean algebra $\mathcal{B}$ constructed in [10] satisfies $irr(\mathcal{B}) < s(K^2_\mathcal{B})$.

In **section 1**, we study the relation between the cardinal invariant $bd(K)$ and $w(K)$, the topological weight of K. We prove the following result