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The spectra of polynomial equations with varying exponents

Asaf Hadari

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A B S T R A C T

Motivated by results of McMullen about the Teichmüller polynomial, we study the dependence of solutions of equations of the form $a_0 + a_1 z^{\ell_1} + \ldots + a_m z^{\ell_m} = 0$, on the exponents $\ell_1, \ldots, \ell_m$. We apply our results to equations that appear in the theory of 3-manifolds fibering over the circle, and the theory of free-by-cyclic groups, and in graph theory. In particular, we provide descriptions of the roots of the Alexander polynomial of a fibered 3-manifold, the Teichmüller polynomials associated to such a manifold or to a free by cyclic group, and the family of characteristic polynomials of a fixed directed graph with varying edge lengths.

1. Introduction

Let M be a hyperbolic 3-manifold that fibers over the circle. Such a manifold may fiber over the circle in many different ways. To each such fibration, one can associate an integral cohomology class $\omega \in H^1(M;\mathbb{Z})$ by pulling back the fundamental class on S^1. Such a cohomology class is said to be *fibered*. In [8], Thurston introduces a norm on $H^1(M;\mathbb{R})$ whose unit ball is a polytope. This polytope has a collection of open, top dimensional faces called *fibered faces*, with the property that the fibered cohomology classes are precisely the primitive integral points in the cones over the fibered faces.

To each fibration we can also associate a fiber Σ_ω and a pseudo-Anosov monodromy $\varphi_\omega \in \mathrm{Mod}(\Sigma_\omega)$. The dependence of φ_ω on ω has been a subject of interest for decades. In [4], Fried shows that if we set $\Lambda(\omega)$ to be the logarithm of the dilatation of φ_ω, then Λ can be extended to a real analytic, degree -1, convex function that goes to ∞ as ω approaches the boundary of a fibered cone.

In [7], McMullen provides an algebraic description for the function Λ. He introduces a polynomial $\Theta = \sum_g a_g g \in \mathbb{Z}[H_1(M,\mathbb{Z})/\text{torsion}]$ called the *Teichmüller polynomial*, such that for any fibered cohomology class ω in a given fibered face, the dilatation of φ_ω is the largest (in absolute value) solution of the equation $\sum a_g t^{\omega(g)} = 0$.

The other solutions of this equation are also geometrically significant. The polynomial $\sum a_g t^{\omega(g)}$ is (up to a cyclotomic factor) the characteristic polynomial of the action of φ_ω on a module called a *train track module*. McMullen is able to analyze the largest of root of this polynomial, but the technology he uses cannot

E-mail address: hadari.asaf@gmail.com.

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be employed to analyze any of the other roots. We will show later on that all roots of this polynomial vary continuously on a fibered cone, and describe their behavior at its boundary.

There is a similar, well known object that can be attached to a fibered 3-manifold that also packages information about the way φ_ω depends on ω: the *Multivariate Alexander polynomial.* This is once again a polynomial $\Delta = \sum b_g g \in \mathbb{Z}[H_1(M,\mathbb{Z})/\text{torsion}]$. It has the property that for any fibered ω, the univariate polynomial $\sum b_g t^{\omega(g)}$ is the characteristic polynomial of the induced action of φ_ω on $H_1(\Sigma_\omega)$.

We will show later in the paper that the roots of the Alexander polynomial also vary continuously, and study where they go to ∞. There are no prior known results describing how the roots of this polynomial depend on ω. Obtaining such results was the original impetus for this paper.

A useful way to think about the Alexander and Teichmüller polynomials is that they describe univariate polynomials whose coefficients are fixed, but whose exponents may vary.

Let $p(z) = a_0 + a_1 z^{\ell_1} + \ldots + a_m z^{\ell_m}$ be a polynomial. It is a classical problem to study the dependence of the roots of p on the coefficients $a = (a_0, \ldots, a_m)$. In this paper we study a related question: how do the roots of p depend on the exponents $\ell = (\ell_1, \ldots, \ell_m)$?

An initial obstruction to studying this question is that it does not make sense for non-integer exponents without choosing a branch of the logarithm function. To avoid this problem, we write $z = e^w$ and turn our attention to studying equations of the form:

$$a_0 + \sum a_i e^{\ell_i w} = 0.$$

We call a function of the form $Q(w,\ell) = a_0 + \sum a_i e^{\ell_i w}$ a *poly-exponential* (these types of expressions are also sometimes called *exponential sums*).

In recent years there have been several additional poly-exponentials in the literature that package information in a similar manner to the Teichmüller polynomial and the Alexander polynomial.

There is a well known idea in low dimensional topology that theorems about mapping class groups often have counterparts in $\mathrm{Out}(F_n)$, the outer automorphism group of a free group. In that setting, there is a construction analogous to a 3-manifold fibering over the circle. Dowdall, Kapovich, Leininger [3] and separately by Algom-Kfir, Hironaka and Rafi [1] discovered analogs for the Teichmüller polynomial in this setting and showed that its largest root has properties similar the properties of the dilatation root mentioned above.

In [6] McMullen introduces a multivariable polynomial called the *Perron polynomial* that describes the following situation. Let Γ be a fixed directed graph. Let ℓ be a function from $E(\Gamma)$ to $\mathbb{N}$. We think of ℓ as assigning integer lengths to the edges of Γ to produce a metric graph Γ_ℓ. This graph can be realized combinatorially by subdividing the edges of Γ. The question McMullen studies is – how do the spectral radii of the characteristic polynomials of the adjacency matrices of Γ_ℓ depend on the choice of ℓ? This is a one dimensional combinatorial analog of the following well known question – how do the eigenvalues of the Laplace–Beltrami operator vary on the moduli space of Riemann surfaces? McMullen shows that these spectral radii are the largest roots of a poly-exponential associated to the Perron polynomial, and is able to show that they share the same properties of the dilatation root mentioned above.

The goal of this paper is to provide tools for studying the roots of poly-exponentials. We use the dilatation function on a fibered cone as a prototype for the kinds of behavior we wish to study. In particular, we address three questions:

(a) Do all roots of a poly-exponential vary continuously with the exponents?
(b) Does the spectral radius of a poly-exponential vary continuously with the exponents?
(c) What is the limiting behavior of roots at the boundaries of their respective cones?