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# Some remarks on the uniqueness of decomposition into Cartesian product

Daria Michalik [a,b,*]

[a] *Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland*
[b] *Faculty of Mathematics and Natural Sciences, College of Science, Cardinal Stefan Wyszyński University, Wóycickiego 1/3, 01-938 Warszawa, Poland*

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ABSTRACT

A decomposition into Cartesian product of prime factors is, in general, not unique. Most of positive results concerning the decomposition uniqueness deal with 1-dimensional spaces. In the paper we give some sufficient conditions on factors for the uniqueness of decomposition to hold without any dimension restrictions.



## 1. Introduction

A space $X$ is called prime if it is not homeomorphic to Cartesian product of two spaces, each containing at least two points. A decomposition into Cartesian product of prime factors is, in general, not unique. The products $[0, 1] \times [0, 1)$ and $[0, 1) \times [0, 1)$ are homeomorphic, but the factors are not homeomorphic. Even in a compact case the decomposition uniqueness does not hold. As was proved by Borsuk [2], there exists a countable family of different continua $\{X_i\}_{i\in\mathbb{N}}$ such that each $X_i$ is AR and $X_i \times I$ are homeomorphic, for every $i \in \mathbb{N}$. There are a few positive results concerning the decomposition uniqueness onto 1-dimensional factors. Recently, in [7], the author proved that the decomposition into finite Cartesian product of locally connected curves is unique. It is natural to ask when the decomposition is unique without any dimensional restrictions.

In this paper we give some sufficient conditions on factors for the decomposition uniqueness to hold. The main tool is based on the concept of homotopic zones of points and homotopically labile and stable points

* Correspondence to: Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland.
*E-mail address:* d.michalik@uksw.edu.pl.

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used already in [5] and [7]. The definition of set $A^X$ and its properties come from [6]. Some ideas of proofs in Section 5 are also adopted from [6].

## 2. Notation and elementary properties

Our terminology follows [3] and [4]. All maps in this paper are continuous.

If $X = X_1 \times \cdots \times X_n$ then for every $i \in \{1, \ldots, n\}$ a map $p_{X_i}\colon X \to X_i$ is a natural projection onto $X_i$ and for every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$ a map

$$r^i_{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)} : X_i \to X$$

is defined by the formula

$$r^i_{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}(x) := (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n).$$

**Definition 1.** A continuous map $h\colon X \times I \to X$ is a *homotopic deformation* if $h(x, 0) = x$ for every $x \in X$.

## 3. Stable, labile and movable points

**Definition 2.** A point $x \in X$ is *stable* if for every homotopic deformation $h$ of space $X$ we have $h(x, 1) = x$.

A point $x \in X$ is *movable* if there exists a homotopic deformation $h$ of space $X$ such that $h(x, 1) \neq x$.

A point $x \in X$ is *labile* if for every $y \in X$ there exists a homotopic deformation $h$ of space $X$ such that $h(x, 1) = y$.

We will denote the set of stable points by $S(X)$, the set of movable points by $P(X)$ and the set of labile points by $R(X)$.

It is obvious, from the definitions, that

$$X = P(X) \cup S(X) \quad \text{and} \quad P(X) \cap S(X) = \emptyset.$$

If $X$ consists of one point then $S(X) = R(X) = X$ and if $X$ contains more than one point then

$$R(X) \subseteq P(X).$$

**Example 1.** If $X$ is a locally connected curve then $R(X)$ is a set of points of $X$ with a neighborhood being a dendrite, $S(X) = X \setminus R(X)$ and $S(X)$ is a set of points of $X$ such that every neighborhood of every point contains a simple closed curve.

For the proof see [1].

**Definition 3.** Let $A \subset X$. The set of points $x$ from $X$ such that for every point $y \in A$ there exists a homotopic deformation $h$ of space $X$ satisfying $h(y, 1) = x$ is denoted by $A^X$.

**Lemma 1.** *Let $h\colon X \to Y$ be a homeomorphic embedding and $A \subset X$. Then*

$$h(A^X) = h(A)^{h(X)}.$$

**Proof.** Let $y_0 \in h(A^X)$ and $h(x_0) = y_0$. Then for every $x \in A$ there exists a homotopic deformation $H_x\colon X \times I \to X$ such that $H_x(x, 1) = h^{-1}(y_0) = x_0$. Let us fix arbitrary $\tilde{y} \in h(A)$. Let us observe that a map $\tilde{H}_{\tilde{y}}\colon h(X) \times I \to h(X)$ defined by the formula: