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Powers of elements of the series substitution group $\mathcal{J}(\mathbb{Z}_2)$

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A R T I C L E I N F O

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For the Jennings group $\mathcal{J}(\mathbf{k})$ of substitutions of formal power series with coefficients in a field $\mathbf{k}$ of positive characteristic (the Nottingham group), the depth of the powers of its elements is studied. In particular, it is shown that the case of a field with characteristic 2 is completely different from the case of a field with odd prime characteristic. It is also shown that the case of the field $\mathbf{k} = \mathbb{Z}_2$ differs from the case of other fields with characteristic 2. Explicit embeddings of the groups $\mathbb{Z}_{p^m}$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$ in $\mathcal{J}(\mathbb{Z}_p)$ are constructed.

1. Introduction

This paper is devoted to a problem of both algebraic and topological origins. Given a group G, we study its power G^n, i.e., the set of all elements of G each of which can be represented as a product of the nth powers of some elements of this group.

Below we cite some results related to the group considered in this paper.

In the 1954 paper [11] Jennings introduced the group $\mathcal{J}(\mathbf{k})$ of all formal power series of the form

$$f(x) = x + \alpha_1 x^2 + \alpha_2 x^3 + \dots = x(1 + \alpha_1 x + \alpha_2 x^2 + \dots), \quad \alpha_n \in \mathbf{k},$$

in a variable x with coefficients in a commutative ring $\mathbf{k}$ with identity. This is a group under the operation of substituting series into series: $f \circ g = f\big(g(x)\big)$.

If the coefficient ring $\mathbf{k}$ is a finite field (of prime characteristic), then the group $\mathcal{J}(\mathbf{k})$ is called the Nottingham group.

The group $\mathcal{J}(\mathbf{k})$ has a natural grading

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$$\mathcal{J}(\mathbf{k}) = \mathcal{J}_1(\mathbf{k}) \supset \cdots \supset \mathcal{J}_d(\mathbf{k}) \supset \mathcal{J}_{d+1}(\mathbf{k}) \supset \dots ,$$

where $\mathcal{J}_d(\mathbf{k}) \subset \mathcal{J}(\mathbf{k})$ consists of all formal series starting with $\alpha_d x^{d+1}$, in which $\alpha_1 = \cdots = \alpha_{d-1} = 0$. The sets $\mathcal{J}_d(\mathbf{k})$ are normal subgroups of $\mathcal{J}(\mathbf{k})$ [7, p. 207]. If $x \neq f(x) \in \mathcal{J}(\mathbf{k})$, then there exists an integer $d \geq 1$ such that $f \in \mathcal{J}_d(\mathbf{k}) \setminus \mathcal{J}_{d+1}(\mathbf{k})$. We refer to this d as the *depth* of f and denote it by $d(f)$. We define the depth of the identity element $(f(x) = x)$ to be ∞.

Jennings showed [11, Theorem 2.5] that, in the Nottingham group,

$$(\mathcal{J}_d(\mathbf{k}))^p \subset \mathcal{J}_{pd}(\mathbf{k}).$$

In 1990 York improved Jennings' result [22, Theorems 6–10]. For a prime $p \geq 3$, Camina described the subgroup $(\mathcal{J}_d(\mathbf{k}))^p$ generated by the pth powers of elements of $\mathcal{J}_d(\mathbf{k})$ [7, Theorem 6].

There is a topological flavor already in the fundamental theorem of algebra. A polynomial $p(z)$ of degree $\deg p = n$ has precisely n roots (counting multiplicities). The fundamental theorem of algebra relates a global property of a polynomial (degree) with its local properties (root multiplicities). The notion of the degree of a polynomial needs no explanation. The notion of the multiplicity μ of a root z_0 (at which $p(z_0) = 0$) is natural as well. Without loss of generality, we can assume that $z_0 = 0$; then the multiplicity μ is the number for which $p(z) = \alpha_n z^n + \cdots + \alpha_\mu z^\mu$, where $\alpha_\mu \neq 0$.

The fundamental theorem of algebra has a far-reaching topological generalization. For a continuous map $f\colon (\mathbb{R}^m, x_0) \to (\mathbb{R}^m, 0)$ such that $f^{-1}(0) = x_0$, the multiplicity $\mu(f, x_0)$ is defined. For a continuous map $f\colon X^m \to Y^m$ of a compact oriented manifold X^m without boundary to a compact oriented manifold Y^m without boundary of the same dimension m, the degree $\deg f$ is defined; for a polynomial map of the Riemann sphere, this degree coincides with its algebraic degree. If, under the above assumptions, the set $f^{-1}(y)$ is finite for some point $y \in Y$, then

$$\deg f = \sum_{x \in f^{-1}(y)} \mu(f, x).$$

There is also fixed point index theory, which is parallel to the theory of image multiplicity. For a self-map $f\colon X \to X$ of a good enough space X (such as a compact manifold or a polyhedron), one of the universal tools for proving the existence of a fixed point is the Hopf–Lefschetz theorem. If the Lefschetz number $\Lambda(f)$ is nonzero, then there is a fixed point. For any isolated fixed point x of a self-map of a polyhedron, an integer $\operatorname{ind}(f, x)$ is defined, which is called the index of this fixed point. The index of an isolated fixed point of $f\colon (\mathbb{R}^m, x_0) \to (\mathbb{R}^m, x_0)$ equals the multiplicity of the zero of the shift map $g = \operatorname{Id} - f\colon (\mathbb{R}^m, x_0) \to (\mathbb{R}^m, 0)$. The Hopf–Lefschetz theorem relates a global property of a map with its local properties. If the fixed point set $\operatorname{Fix} f$ is finite, then

$$\Lambda(f) = \sum_{x \in \operatorname{Fix} f} \operatorname{ind}(f, x).$$

Thus, for any positive integer n such that $\operatorname{Fix} f^n$ is finite, we have

$$\Lambda(f^n) = \sum_{x \in \operatorname{Fix} f^n} \operatorname{ind}(f^n, x).$$

The behavior of the numbers $\Lambda(f^n)$ is determined by the induced homomorphism of the (co)homology groups of the space X. Studying the sequence of integers $\{\operatorname{ind}(f^n, x)\}_{n=1}^\infty$ for a fixed point x, we can potentially prove the presence of periodic points.

Steinlein [21] and Zabreiko with Krasnosel'skii [23] showed independently that, for any fixed point of a map f isolated in the fixed point set of any iteration (power) f^n of f, there exists a sequence of integers (multiplicities) $\{a_d\}_{d=1}^\infty$ such that, for any positive integer n, we have