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Coincidence points in the cases of metric spaces and metric maps

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A B S T R A C T

In the first half of the paper, we are concerned with the problems of existence (and searching) of coincidence points and the common preimage of a closed subset (in particular, a common root) in the case of a finite system of mappings of one metric space to another one. The second half of the paper is devoted to fiberwise variants of Arutyunov's theorem on coincidence points.

Obtaining the main results of the paper is based on the use of the class of almost exactly (α, β)-search functionals that is wider than Fomenko's class of (α, β)-search functionals.

0. Introduction

We will use 1) "space" instead of "metric space" in Sections 1 and 2 and instead of "topological space" in Section 3; 2) "map" instead of "continuous mapping". For a metric space (X, ρ) and $x \in X$, by $O_r x \equiv O(x, r)$ (more detailed, $O_r^X x \equiv O^X(x, r)$) denote the r-neighborhood of x in X. If we use B instead of O in this notation, we will get the notation for the closed ball with the center x of radius r. For metric spaces (X, ρ_X) and (Y, ρ_Y), we consider only the metric $\rho_X + \rho_Y$ on $X \times Y$. Note also that for a mapping $f\colon X \to Y$ and $fX \subset Y' \subset Y$, the corestriction $cor_{Y'} f$ of f to Y' is the mapping of X to Y' such that $(cor_{Y'} f)(x) = fx$ for any $x \in X$. If $Y' = fX$ then $cor\, f$ is used instead of $cor_{fX} f$.

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In Sections 1 and 2, we consider the problems of the existence and searching of coincidence points and the common preimage of a closed subset (in particular, the common root) in the case of a finite system of mappings of one metric space to another one.

In [7], T.N. Fomenko used (α, β)-search functionals (i.e. mappings to $[0, +\infty)$) on metric spaces to solve the problems mentioned above. An (α, β)-search functional φ on a metric space X allows to obtain, for any $x \in X$, a fundamental sequence of points $x_0 = x, x_1, \ldots, x_n, \ldots$ in X such that $\varphi(x_n) \xrightarrow{n\to\infty} 0$. Under some additional conditions (for example, if X is complete and φ is continuous), there exists $\xi = \lim\limits_{n\to\infty} x_n \in X$ such that

$$\varphi(\xi) = 0 \quad \text{and} \quad \rho(x, \xi) \leqslant \frac{\varphi(x)}{\alpha - \beta}.$$

(In particular:

1. for a mapping $f\colon X \to X$ and $\varphi(x) = \rho(x, f(x))$, $x \in X$, the equality $\varphi(\xi) = 0$ means that ξ is a fixed point for f;
2. for mappings $f, g\colon X \to Y$ and $\varphi(x) = \rho(f(x), g(x))$, $x \in X$, the equality $\varphi(\xi) = 0$ means that ξ is a coincidence point for f and g.)

In Section 1, the class of almost exactly (α, β)-search functionals on metric spaces is defined. This class is wider than the class of (α, β)-search functionals. It is used in Section 2 to obtain (in more general situations) results that are similar to ones in [7].

In Section 3, a fiberwise variants of Arutyunov's theorem on coincidence points ([2], Theorem 1) are obtained. The proofs of our theorems are based on the use of almost exactly (α, β)-search functionals.

We consider only one-valued mappings.

1. Search functionals on metric spaces

Fix a space (X, ρ).

Let $F(X)$ be the set of all (not necessarily continuous) mappings of X to itself and $CF(X)$ the set of all continuous mappings of X to itself. For $A, B \in F(X)$, set

$$d_F(A, B) = \sup\{\rho(Ax, Bx)\colon x \in X\}.$$

For $A \in F(X)$ and $C \in CF(X)$, set $F(X, A) = \{B \in F(X)\colon d_F(A, B) < +\infty\}$ and $CF(X, A) = \{D \in CF(X)\colon d_F(C, D) < +\infty\}$. It is evident that:

for $A, B \in F(X)$, either $F(X, A) \cap F(X, B) = \varnothing$ or $F(X, A) = F(X, B)$ and this equality is equivalent to the inequality $d_F(A, B) < +\infty$;

for $C, D \in CF(X)$, either $CF(X, C) \cap CF(X, D) = \varnothing$ or $CF(X, C) = CF(X, D)$ and this equality is equivalent to the inequality $d_F(C, D) < +\infty$;

for every $A \in F(X)$ (respectively, $A \in CF(X)$), the function d_F is a metric on $F(X, A)$ (respectively, on $CF(X, A)$).

The sets of type $F(X, A)$ (respectively, $CF(X, A)$) will be called *metric parts* of $F(X)$ (respectively, of $CF(X, A)$).

It is easy to see that every metric part of $F(X)$ (respectively, of $CF(X)$) is a complete space (with the metric d_F) if the space X is complete.

Recall the definition of (α, β)-search functionals given by T.N. Fomenko in [7].

Further, let $\mathbb{R}_+ = \{x \in \mathbb{R}\colon x \geqslant 0\}$ and $\mathbb{N}_+ = \{0\} \cup \mathbb{N}$.