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Quasirational relation modules and p-adic Malcev completions

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ABSTRACT

We introduce the concept of quasirational relation modules for discrete (pro-p) presentations of discrete (pro-p) groups. It is shown, that this class of presentations for discrete groups contains CA-presentations and their subpresentations. For pro-p-groups we see that all presentations of pro-p-groups with a single defining relation are quasirational. We offer definitions of p-adic $G(p)$-completion and p-adic rationalization of relation modules which are adjusted to quasirational pro-p-presentations. p-adic rationalizations of quasirational relation modules of pro-p-groups are isomorphic to $\mathbb{Q}_p$-points of abelianized p-adic Malcev completions.

1. Introduction and motivations

Whitehead asphericity question is one of the oldest problems in Combinatorial Group Theory. Similarly, validity of analogs of Lyndon Identity Theorem in Combinatorial Theory of pro-p-Groups [14, 10.2] is still far from being understood. We introduce a class of finite presentations ("quasirational" presentations) which contains aspherical presentations as well as their subpresentations. In a pro-p case (pro-p-groups are projective limits of finite p-groups with their pro-p-presentations, every pro-p-group could be presented as some factor of free pro-p-group with an appropriate space of topological generators by a certain closed normal subgroup (see [16])) our concept includes one-relator pro-p-groups.

Permutational features of relation modules play the key role in asphericity type problems [3,11]. Moreover permutationality unlike asphericity behaves properly with respect to coinvariant completions of relation modules and holds by their scalar extensions. We will see that quasirational presentations may be studied by passing to rationalized completions $\overline{R}\widehat{\otimes}\mathbb{Q}_p := \varprojlim R/[R, R\mathcal{M}_n] \otimes \mathbb{Q}_p$ (since $\varprojlim$ is left exact for quasirational pro-p-presentations we have an embedding of abelian groups $\overline{R} \hookrightarrow \overline{R}\widehat{\otimes}\mathbb{Q}_p$) in a spirit of Gaschütz theory (see [5]). $\overline{R}\widehat{\otimes}\mathbb{Q}_p$ has a structure of topological $\mathcal{O}(F_u)^*$-module in a sense of [6], where F_u is a free prounipotent group with a complete Hopf algebra $\mathcal{O}(F_u)^* := Hom_k(\mathcal{O}(F_u), k)$ (the "coordinate ring" of F_u i.e. dual to the

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representing Hopf algebra $\mathcal{O}(F_u)$ of F_u [18, 3]). The same structure is valid for quasirational presentations of pro-p-groups.

As we will show, there is an isomorphism $\overline{R}\widehat{\otimes}\mathbb{Q}_p \cong \overline{R_u^\wedge}(\mathbb{Q}_p)$ in the category of topological $\mathcal{O}(F_u)^*$-modules, where $\overline{R_u^\wedge}(\mathbb{Q}_p)$ is a certain prounipotent module over the same complete Hopf algebra. The latter category seems extremely convenient for the required calculations.

2. Quasirational relation modules

For pro-p-groups, fix a prime $p > 0$ throughout the paper (see [16] for details on pro-p-groups). **For discrete groups, p will vary**. Let G be a (pro-p)group which has a (pro-p)presentation of finite type

$$1 \to R \to F \to G \to 1 \tag{1}$$

Let $\overline{R} = R/[R, R]$ be the corresponding relation G-module, where $[R, R]$ is a (closed) commutator subgroup (in the pro-p-case). Then denote $\mathcal{M}_n$ the corresponding Zassenhaus p-filtration of F, which is defined by the rule $\mathcal{M}_n = \{f \in F \mid f - 1 \in \Delta_p^n,\ \Delta_p = ker\{\mathbb{F}_pF \to \mathbb{F}_p\}\}$ (see [9, 7.4] for details).

Definition 1. A presentation (1) is **quasirational** if for every $n > 0$ and each prime $p > 0$ the $F/R\mathcal{M}_n$-module $R/[R, R\mathcal{M}_n]$ has no p-torsion (p is fixed for pro-p-groups and run all primes $p > 0$ and corresponding p-Zassenhaus filtrations in discrete case). The relation modules of such presentations will be called **quasirational relation modules**.

Example 1. Let $\overline{R}$ be a (pro-p-)permutational G-module, so by definition $\overline{R}$ has a permutational G-basis as a projective $(\mathbb{Z}_p)\mathbb{Z}$-module. Then, obviously, $\overline{R}$ is quasirational.

Remark 1. The definition of quasirationality is rigid. Since $R/[R, F]$ has no torsion, then the Schur multiplicator has no torsion. As a consequence we see that $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ has no quasirational presentations. If G is a finite p-group, then $H_2(G)$ has a finite exponent and is not trivial while G is not a cyclic (since its $mod(p)$ factor is not trivial). Hence the only finite p-groups, which have quasirational presentations are cyclic. This shows that quasirational finite p-groups match with finite aspherical pro-p-groups [11, Theorem 2.7].

We need the following elementary

Lemma 1. *Let G be a finite p-group acting on a finite abelian group M of exponent p. Then the factor module of coinvariants $M_G = M/(g-1)M \neq 0$, where $(g-1)M$ is a submodule of M, generated by elements of the form $(g-1)m$, where $g \in G, m \in M$.*

Proof. We prove by induction on a rank n of M. If $n = 1$ then $M = \mathbb{Z}/p\mathbb{Z}$ is a trivial G-module, since $(\mid Aut(\mathbb{Z}/p\mathbb{Z}) \mid, p) = (\mid \mathbb{Z}/(p-1)\mathbb{Z} \mid, p) = 1$, so $M_G = M \neq 0$. Let $n = k$, then a submodule of G-fixed elements of M is not trivial [15, Ch IX, 1] and contains $M_0 = \mathbb{Z}/p\mathbb{Z}$ which has trivial G-action. Let $M_1 = M/M_0$ and $\psi : M \to M_1$ is the corresponding homomorphism of factorization. Since $\psi((g-1)M) = (g-1)M_1$, ψ induces the surjection $M_G \twoheadrightarrow (M_1)_G$. But $(M_1)_G \neq 0$ by induction hence $M_G \neq 0$. □

There is an old problem due to [14, 10.2] concerning the description of relation modules of pro-p-groups with a single defining relation (i.e. $dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$ [16, 4.3]). We have:

Proposition 1. *Suppose (1) is a presentation of a pro-p-group G with a single defining relation, then (1) is quasirational.*