Topology and its Applications 201 (2016) 92–109

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Dimension and universality on frames

S.D. Iliadis

Department of General Topology and Geometry, Faculty of Mechanics and Mathematics, Moscow State University (M.V. Lomonosov), Moscow 119991, Russia

ARTICLE INFO

Article history:
Received 31 December 2014
Accepted 8 June 2015
Available online 22 December 2015

MSC:
06A06
06D22
06F30
54H12
54F05

Keywords:
Lattice
Frame
Dimension on frames
Universal frame
Saturated class of frames
Decomposition invariant

ABSTRACT

In this paper we first introduce the notion of a saturated class of frames and then define a dimension-like invariant of a frame, the so-called decomposition invariant denoted by *dec*, which is a cardinal (in particular maybe a non-negative integer), and using the notion of the saturated class prove that in the class of all frames of weight less than or equal to a fixed infinite cardinal τ having decomposition invariant $\leq \mu$, where μ is a fixed cardinal less than or equal to τ, there are universal elements.

1. Introduction

The authors in [4] formulated the following results given in the frame version:

(a) For every completely regular frame L there exists a homomorphism of $\mathbb{I}^{w(L)}$ onto L. (Equivalently, given a cardinal κ, $\mathbb{I}^{\kappa}$ is universal in the class of all completely regular frames of weight smaller than or equal to κ.)

(b) For every regular frame L with a countable base there exists a homomorphism of $\mathbb{I}^{\omega}$ onto L. (Equivalently, $\mathbb{I}^{\omega}$ is universal in the class of all regular frames with a countable base.)

(c) For a zero-dimensional frame L there exists a homomorphism of $\mathbb{B}_4^{w(L)}$ onto L. (Equivalently, given a cardinal κ, $\mathbb{B}_4^{\kappa}$ is universal in the class of all zero-dimensional frames of weight smaller than or equal to κ.)

On behalf of the authors of the paper [5] the author of the present paper would like to note the following. Unfortunately, the authors of the papers [5] and [7] were not aware of the paper [4] when these papers

E-mail address: s.d.iliadis@gmail.com.

http://dx.doi.org/10.1016/j.topol.2015.12.029
0166-8641/

were written and we would like to thank Javier Gutiérrez García for addressing our attention to the paper [4]. This is also the reason why in the paper [5] (published later than [4]) "open problems" were posed, whose answers are exactly the above mentioned results: the result (*a*) for the Problem 2.4(1), (*b*) for the Problem 2.4(2) in the particular case $\tau = \omega$, and the result (*c*) for the Problem 2.4(3). The same is true why in the paper [7] the above result (*a*) is proved. Although, it is necessary to note that the results of [5] and [7] are proved by completely different ways without the use of separating families and product of locales (frames) as it is done in the papers [4,8,10].

For the construction of classes of spaces having universal elements the different dimension-like invariants play an important role. Many such classes are defined as classes of spaces for which a given dimension-like invariant takes some concrete value. However, the dimension theory for frames is not sufficiently developed. Although for the frames some dimension invariants are defined (see for example [2,3,12]) they were not used to define classes of frames with universal elements. (The exception is the class of zero-dimensional spaces.)

In the present paper first we introduce the notion of a saturated class of frames. Such classes are **saturated** by universal elements. The class of all frames of a given weight is saturated. In a forthcoming paper we shall prove that the class of all regular and the class of all completely regular frames of a given weight are also saturated. Saturated classes have the following important property: for a given cardinal τ the intersection of not more than τ many saturated classes of frames of weight not more than τ is also a saturated class. This property can be used for the construction of new classes of frames with universal elements from known classes having universal elements. For example, if we have proved that in the class of all frames of a given weight and a given dimension-like invariant there are universal elements proving that this class is saturated, then it is not necessary to prove the same result for regular or for completely regular frames. These results follow from the above property of saturated classes. Saturated classes play an important role in the proofs of theorems concerning the existence of universal elements.

Furthermore we define a dimension-like invariant of a frame F, called decomposition invariant and denoted by $dec(F)$, which is an integer or cardinal. If $dec(F) = n \in \omega$, then F is called n-dimensional and if $dec(F) = \omega$ then it is called countable-dimensional. It is proved that $dec(F) = 0$ if and only if F is zero-dimensional (see [1]). We prove that the class of all frames F of weight $\leq \tau$ with $dec(F) \leq \mu$, where μ and τ, $\mu \leq \tau$, are fixed cardinals, is a saturated class which means that in this class there are universal elements.

2. Preliminaries

2.1. Definitions and notation. An *ordinal number* is the set of smaller ordinal numbers, and a *cardinal number* is an initial ordinal number. Therefore, for ordinals δ and τ relations $\delta \in \tau + 1$ and $\delta \leq \tau$ are equivalent. By ω we denote the least infinite cardinal and by τ **a fixed infinite cardinal**. The elements of τ will be denoted by δ, ε, η, ζ, and ξ. The letters θ, ϑ, and φ (possibly with some indexes) are used for the notation of mappings of indices. For a set X we denote by $\mathcal{F}(X)$ the set of all **non-empty** finite subsets of X. The set $\mathcal{F}(\tau)$ will be denoted simply by $\mathcal{F}$. The elements of $\mathcal{F}$ will be denoted by s, p, and t (possibly with some indexes). The symbol $\equiv$ in a relation means that one or both sides of the relation are new notations.

Each mapping f of a set κ onto a set B is called a *κ-indexation* (or simply an *indexation*) of X and will be denoted by $B = \{a_\delta : \delta \in \kappa\}$ where $a_\delta = f(\delta)$, $\delta \in \kappa$.

Recall that a *frame* is a complete lattice L in which

$$x \wedge \sup(S) = \sup\{x \wedge y \colon y \in S\}$$

for any $x \in L$ and any $S \subseteq L$. We denote the top element and the bottom element of L by 1_L and 0_L respectively. For basic information about frames or pointfree topology see [11] or [9].