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Verbal covering properties of topological spaces ☆

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ABSTRACT

For any topological space X we study the relation between the universal uniformity $\mathcal{U}_X$, the universal quasi-uniformity $q\mathcal{U}_X$ and the universal pre-uniformity $p\mathcal{U}_X$ on X. For a pre-uniformity $\mathcal{U}$ on a set X and a word v in the two-letter alphabet $\{+,-\}$ we define the verbal power $\mathcal{U}^v$ of $\mathcal{U}$ and study its boundedness numbers $\ell(\mathcal{U}^v)$, $\bar{\ell}(\mathcal{U}^v)$, $L(\mathcal{U}^v)$ and $\bar{L}(\mathcal{U}^v)$. The boundedness numbers of (the Boolean operations over) the verbal powers of the canonical pre-uniformities $p\mathcal{U}_X$, $q\mathcal{U}_X$ and $\mathcal{U}_X$ yield new cardinal characteristics $\ell^v(X)$, $\bar{\ell}^v(X)$, $L^v(X)$, $\bar{L}^v(X)$, $q\ell^v(X)$, $q\bar{\ell}^v(X)$, $qL^v(X)$, $q\bar{L}^v(X)$, $u\ell(X)$ of a topological space X, which generalize all known cardinal topological invariants related to (star-)covering properties. We study the relation of the new cardinal invariants ℓ^v, $\bar{\ell}^v$ to classical cardinal topological invariants such as Lindelöf number ℓ, density d, and spread s. The simplest new verbal cardinal invariant is the foredensity $\ell^-(X)$ defined for a topological space X as the smallest cardinal κ such that for any neighborhood assignment $(O_x)_{x\in X}$ there is a subset $A \subset X$ of cardinality $|A| \le \kappa$ that meets each neighborhood O_x, $x \in X$. It is clear that $\ell^-(X) \le d(X) \le \ell^-(X)\cdot\chi(X)$. We shall prove that $\ell^-(X) = d(X)$ if $|X| < \aleph_\omega$. On the other hand, for every singular cardinal κ (with $\kappa \le 2^{2^{\mathrm{cf}(\kappa)}}$) we construct a (totally disconnected) T_1-space X such that $\ell^-(X) = \mathrm{cf}(\kappa) < \kappa = |X| = d(X)$.

0. Introduction

In this paper we suggest a uniform treatment of many (star-)covering properties considered in topological literature. Namely, for every word v in the two-letter alphabet $\{+,-\}$ we define v-compact, weakly v-compact, v-Lindelöf, and weakly v-Lindelöf spaces, and the corresponding cardinal topological invariants L^v, $\bar{L}^v$, ℓ^v, $\bar{\ell}^v$, which generalize many known cardinal invariants that have covering nature. In particular, ℓ^+ and $\bar{\ell}^+$ coincide with the Lindelöf and weakly Lindelöf numbers, ℓ^{-+} coincide with the weak extent and ℓ^{+-+} coincides with the star-Lindelöf number. The cardinal characteristics L^v, $\bar{L}^v$, ℓ^v, $\bar{\ell}^v$ are defined in

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Section 4. Sections 1, 2 are of preliminary character and collect known information on covering and star-covering properties in topological spaces and pre-uniform spaces. In Section 3 we introduce three canonical pre-uniformities $p\mathcal{U}_X$, $q\mathcal{U}_X$, $\mathcal{U}_X$ on a topological space X and study the inclusion relation between these pre-uniformities and their verbal powers. Section 5 is devoted to studying the interplay between the density d and the cardinal invariant ℓ^- called the foredensity.

1. Preliminaries

In this section we recall some information on covering properties of topological spaces.

Let ω denote the set of all finite ordinals and let $\mathbb{N} = \omega \setminus \{0\}$ be the set of natural numbers.

For a subset A of a topological space X by $\overline{A}$ we denote the closure of the set A in X.

We recall that a family $(A_i)_{i\in I}$ of subsets of a topological space X is *discrete* if each point $z \in X$ has a neighborhood that meets at most one set A_i, $i \in I$.

1.1. Classical cardinal invariants in topological spaces

We recall that for a topological space X its *character* $\chi(X)$ is defined as the smallest cardinal κ such that each point $x \in X$ has a neighborhood base $\mathcal{B}_x$ of cardinality $|\mathcal{B}_x| \le \kappa$.

Next, we recall the definitions of the basic cardinal invariants composing the famous Hodel's diagram [11, p. 15] (see also [9, p. 225]). For a topological space X let

- $w(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a base of the topology of $X\}$ be the *weight* of X;
- $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a network of the topology of $X\}$ be the *network weight* of X;
- $d(X) = \min\{|A| : A \subset X,\ \overline{A} = X\}$ be the *density* of X;
- $hd(X) = \sup\{d(Y) : Y \subset X\}$ be the *hereditary density* of X;
- $l(X)$, the *Lindelöf number* of X, be the smallest cardinal κ such that each open cover $\mathcal{U}$ of X has a subcover $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| \le \kappa$;
- $hl(X) = \sup\{l(Y) : Y \subset X\}$ be the *hereditary Lindelöf number* of X;
- $s(X) = \sup\{|D| : D$ is a discrete subspace of $X\}$ be the *spread* of X;
- $e(X) = \sup\{|D| : D$ is a closed discrete subspace of $X\}$ be the *extent* of X;
- $c(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a disjoint family of non-empty open sets in $X\}$ be the *cellularity* of X.

These nine cardinal characteristics compose the Hodel diagram [11] in which an arrow $f \to g$ indicates that $f(X) \le g(X)$ for any topological space X. The same convention concerns all other diagrams drawn in this paper.

$$\begin{tikzcd}
 & & w & & \\
 & & nw \arrow[u] & & \\
 & hd \arrow[ur] & & hl \arrow[ul] & \\
d \arrow[ur] & & s \arrow[ul] \arrow[ur] & & l \arrow[ul] \\
 & c \arrow[ul] \arrow[ur] & & e \arrow[ul] \arrow[ur] &
\end{tikzcd}$$