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Game theoretic approach to skeletally Dugundji and Dugundji spaces

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ABSTRACT

Characterizations of skeletally Dugundji spaces and Dugundji spaces are given in terms of club collections, consisting of countable families of co-zero sets. For example, a Tychonoff space X is skeletally Dugundji if and only if there exists an additive c-club on X. Dugundji spaces are characterized by the existence of additive d-clubs.

1. Introduction

The aim of this paper is to provide characterizations of Dugundji and skeletally Dugundji spaces in terms of game theory. Recall that Dugundji spaces were introduced in [10] and their topological description as compacta with a multiplicative lattice of open maps was obtained in [11] and [12]. Skeletally Dugundji spaces are skeletal analogue of Dugundji spaces. Our starting point was spectral characterization of Dugundji spaces revealed the possibility of the usage of inverse systems, see [3]. In [9, Theorem 3.3] it is given several conditions equivalent to skeletally Dugundji spaces. For the purposes of this paper we propose the following one: A Tychonoff space is skeletally Dugundji if it has a multiplicative lattice of skeletal maps, see [9]. So, this paper is a continuation of the topics initiated in [8] and [9].

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Our key notion of additive clubs consisting of co-zero sets was inspired by condition (6) from [4, Theorem 5.3.9], which characterizes weakly projective Boolean algebras (this condition was attributed to T. Jech). Together with methods developed in [6] this allows to expand the study of weakly projective Boolean algebra (or equivalently its Stone spaces) to much wider classes, namely Tychonoff spaces. Another general idea we follow is the possibility to describe some Tychonoff spaces using inverse systems consisting of "nice" bonding maps and second countable spaces. This method is used in papers [6–8], where "nice" maps are surjective skeletal, open or d-open, and inverse systems are constructed by altering well matched collections of countable families of open sets, which are called club families. For a certain club family of open sets one can associate a version of the open–open game. This procedure was initiated in [1], where the existence of an appropriate club family is equivalent to the existence of a winning strategy for player I in the open–open game. It turns out that proofs from [1,6–8,14] dared us to introduce the concept of additive club collection.

The open–open game was introduced by P. Daniels, K. Kunen and H. Zhou [1]. If a space X is Tychonoff, then we can assume that two players are playing with co-zero sets. Thus, there are two players who take turns playing: a round consists of player I choosing a nonempty set $U \in coZ(X)$ and player II choosing a nonempty $V \in coZ(X)$ with $V \subseteq U$. Player I wins if the union of II's open sets is dense in X, otherwise player II wins. In the paper [1], there are a few equivalent descriptions of the open–open game. According to [1, Theorem 1.6], X is I-favorable if and only if the family

$$\{\mathcal{P} \in [coZ(X)]^\omega : \mathcal{P} \subset_c coZ(X)\}$$

contains a club, here $\mathcal{P} \subset_c coZ(X)$ means that $\mathcal{P}$ is completely embedded in $coZ(X)$, compare [1, p. 208]. In fact, this theorem provides a strategy how player I, knowing the club family, can win the open–open game, and vice versa. Because of this interpretation, the question, whether player I can use a club family with some additional features, arises naturally. We describe the cases when player I constructs a lattice of skeletal (open or d-open) maps despite the choices of player II bothers.

If $\mathcal{A}$ is a family of sets, then $\langle\mathcal{A}\rangle$ denotes the least family which contains $\mathcal{A}$ and it is closed under finite intersections and finite unions. By spaces and maps we mean, respectively, infinite Tychonoff spaces and continuous maps. In accordance with J. Mioduszewski and L. Rudolf [5], a map $f : X \to Y$ is said to be *skeletal* (resp., *d-open* in the sense of M. Tkachenko, [13]) if $\operatorname{Int} \overline{f[U]} \neq \varnothing$ (resp., $f[U] \subseteq \operatorname{Int} \overline{f[U]}$) for every nonempty open $U \subseteq X$. Obviously, every d-open map is skeletal. Moreover, any d-open map between compact Hausdorff spaces is always open, see [13]. We say that two maps $f : X \to Y$ and $g : X \to Z$ are *homeomorphic* if there is a homeomorphism $h : Z \to Y$ such that $f = h \circ g$. Let us also mention that weight of a given space X, denoted by $\mathrm{w}(X)$, is always an infinite cardinal. Also, a diagonal product of maps will be briefly called a diagonal. We recommend the book [2] and the paper [1] for undefined notions.

2. Maps constructed using the property *Seq*

Let $\mathcal{P}$ be a family of open subsets of a topological space X. For every $x \in X$ consider the set

$$[x]_{\mathcal{P}} = \{y \in X : y \in V \Leftrightarrow x \in V \text{ for all } V \in \mathcal{P}\}.$$

Let $X/\mathcal{P}$ be the family of all classes $[x]_{\mathcal{P}}$ and $q_{\mathcal{P}} : X \to X/\mathcal{P}$ be the map $x \mapsto [x]_{\mathcal{P}}$. The topology on $X/\mathcal{P}$ is generated by the sets $q_{\mathcal{P}}[V] = \{[x]_{\mathcal{P}} : x \in V\}$, where $V \in \mathcal{P}$. From now on we will assume that $X = \bigcup \mathcal{P}$. So, we have the following fact, see [6, Lemma 1].

Lemma 1. *If $\mathcal{P}$ is closed under finite intersections, then the family $\{q_{\mathcal{P}}[V] : V \in \mathcal{P}\}$ is a base for $X/\mathcal{P}$ and the map $q_{\mathcal{P}}$ is continuous.*