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Products of derived structures on topological spaces

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We consider topological spaces X equipped with an algebra $\mathcal{A}$ of subsets of X and an ideal $\mathcal{I}$ of $\mathcal{A}$. Motivated by the example of the Jordan measurable subsets of $\mathbb{R}$, we consider the *derived structure* obtained by replacing $\mathcal{A}$ by the algebra $\partial\mathcal{A} = \{E \in \mathcal{A} : \partial E \in \mathcal{I}\}$ of sets with negligible boundaries, and $\mathcal{I}$ by $\partial I = \mathcal{I} \cap \partial\mathcal{A}$. In a previous paper by M.R. Burke et al. (2012) [7], the authors classified these derived structures (under some assumptions) and computed densities for them. In the present paper, we extend that work in the context of products of derived structures. We study in greater detail the box cross product $\gamma \boxtimes \delta$ of two set maps $\gamma \in \mathcal{P}(X)^{\mathcal{A}}$, $\delta \in \mathcal{P}(Y)^{\mathcal{B}}$ introduced in joint work of the authors with K. Musiał (2009) [6], examining when it preserves densities and other types of liftings. For preservation of monotonicity, we introduce a variation on the localization property of ideals which is well-known for the meager ideal. An examination of skew products provides a class of structures to which our results apply.

1. Introduction

We consider structures $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I})$, where $(X, \mathfrak{S})$ is a topological space, $\mathcal{A}$ is an algebra of subsets of X and $\mathcal{I}$ is a proper ideal of $\mathcal{A}$. For example, the real line with its usual topology, equipped with the algebra of sets having the property of Baire and the ideal of meager sets, or equipped with the algebra of Lebesgue measurable sets, and the ideal of sets of measure zero. We denote by BQ the class of all such structures

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("basic quadruples"). In [7], motivated by the example of the algebra of *Jordan measurable* subsets of $\mathbb{R}$, i.e., those subsets of $\mathbb{R}$ whose boundaries have measure zero, we defined the *derived structure*

$$\partial(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) = (X, \mathfrak{S}, \partial\mathcal{A}, \partial\mathcal{I}),$$

where $\partial\mathcal{A} := \{E \in \mathcal{A} : \partial E \in \mathcal{I}\}$, $\partial\mathcal{I} := \partial\mathcal{A} \cap \mathcal{I}$. As was shown in [7], some assumptions need to be made in order for derivation to behave well. The natural setting for considering derived structures seems to be the class FQ $\subseteq$ BQ of structures ("fundamental quadruples") $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I})$ satisfying the mild completeness assumption that the boundary ∂E of a set $E \in \mathcal{A}$ belongs to the ideal $\mathcal{I}$ whenever it is contained in a member of $\mathcal{I}$.[2] In [7], under various hypotheses, the structures in FQ were classified and properties of densities and other selectors for the classes of $\mathcal{A}$ modulo $\mathcal{I}$ were established. In particular, the following proposition was proven.

Proposition 1.1. *([7], Proposition 3.4, Corollary 3.7, Corollary 3.12) The class* FQ *is closed under derivation, and derivation is idempotent on* FQ*. For $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ FQ, $\partial\mathcal{A}$ is an algebra closed under the boundary operation and $\partial\mathcal{I} = \{A \in \mathcal{A} : \mathrm{cl}\, A \in \mathcal{I}\}$.*

We write DQ for the class of all derived structures $\partial(X, \mathfrak{S}, \mathcal{A}, \mathcal{I})$ for $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ FQ. The structures $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ DQ can be characterized as follows.

Proposition 1.2. *([7], Corollary 3.14) Let $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ BQ. Then $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ DQ if and only if $\partial E \in \mathcal{I}$ whenever $E \in \mathcal{A}$, i.e., $\partial\mathcal{A} = \mathcal{A}$.*

For a topological space $(X, \mathfrak{S})$, let $\mathcal{A}(X)$ be the *complete basic ring* of X, i.e., the algebra of sets which are equal to an open set modulo the ideal $\mathcal{N}(X)$ of nowhere dense sets. This algebra was introduced by M.H. Stone ([20], Definition 10). Some relevant results of [20] are summarized in the introduction to [7]. We note here only that $\mathcal{A}(X) = \{E \in \mathcal{P}(X) : \partial E \in \mathcal{N}(X)\}$. The structures $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ DQ for which the condition $\mathfrak{S} \cap \mathcal{I} = \{\emptyset\}$, which we denote $[N_\mathcal{I}]$, holds were characterized in [7] in terms of $\mathcal{A}(X)$ as follows. $\mathcal{M}(X)$ denotes the ideal of meager sets of X.

Theorem 1.3. *([7], Theorem 3.16, Corollary 3.11) Let $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ BQ. The following statements are equivalent.*

(i) *$[N_\mathcal{I}]$ is satisfied and $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ DQ.*
(ii) *$\mathcal{A}$ is a subalgebra of $\mathcal{A}(X)$ closed under the boundary operation and $\mathcal{I} = \mathcal{A} \cap \mathcal{N}(X)$.*

When these conditions hold, and X is a Baire space, we also have $\mathcal{I} = \mathcal{A} \cap \mathcal{M}(X)$.

We are interested in when our structures have a lifting (a homomorphism $\mathcal{A} \to \mathcal{A}$ which induces a selector for the equivalences classes $\mathcal{A}/\mathcal{I} \to \mathcal{A}$) or a weaker type of selector. A map $\gamma : \mathcal{A} \to \mathcal{A}$ is a *primitive lifting* if it is a selector for the equivalences modulo $\mathcal{I}$ and fixes $\emptyset$ and X. In other words: $\gamma(\emptyset) = \emptyset$; $\gamma(X) = X$; for all $A \in \mathcal{A}$, $\gamma(A) =_\mathcal{I} A$; and for all $A, B \in \mathcal{A}$, $A =_\mathcal{I} B$ implies $\gamma(A) = \gamma(B)$. A primitive lifting is *monotone* if $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$; *orthogonal* if $A \cap B = \emptyset$ implies $\gamma(A) \cap \gamma(B) = \emptyset$; a *density* if $\gamma(A \cap B) = \gamma(A) \cap \gamma(B)$. (Note that densities are both monotone and orthogonal.) A primitive lifting is

[2] Alternatively, define $\widehat{\partial}\mathcal{A} := \{A \in \mathcal{A} : \partial A \in \widehat{\mathcal{I}}\}$, $\widehat{\partial}\mathcal{I} := \mathcal{I} \cap \widehat{\partial}\mathcal{A}$, $\widehat{\partial}(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) := (X, \mathfrak{S}, \widehat{\partial}\mathcal{A}, \widehat{\partial}\mathcal{I})$, where $\widehat{\mathcal{I}}$ is the completion of $\mathcal{I}$. $\widehat{\partial} = \partial$ on the class FQ (and only there). $\widehat{\partial}$ has the advantage that for any $(X, \mathfrak{S}, \mathcal{A}, \mathcal{I}) \in$ BQ, $\widehat{\partial}\mathcal{A}$ is an algebra, but it still exhibits pathological behavior outside the class FQ, for example, the algebra $\widehat{\partial}\mathcal{A}$ might not be closed under the boundary operation, and the operation $\widehat{\partial}$ on the class BQ is not idempotent.