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Coincidence points principle for set-valued mappings in partially ordered spaces

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ABSTRACT

In the paper the concept of covering (regularity) for set-valued mappings in partially ordered spaces is introduced. The coincidence points problem for set-valued mappings in partially ordered spaces is considered. Sufficient conditions for the existence of coincidence points of isotone and orderly covering set-valued mappings are obtained. It is shown that the known theorems on coincidence points of covering and Lipschitz mappings in metric spaces can be deduced from the obtained results.

Given nonempty sets X, Y and set-valued mappings $\Psi, \Phi : X \rightrightarrows Y$, a point $x \in X$ that satisfies the relation

$$\Psi(x) \cap \Phi(x) \neq \emptyset$$

is called a coincidence point of the mappings Ψ and Φ. In this paper we introduce sufficient conditions for the existence of coincidence points of two set-valued mappings.

In the case when X and Y are metric spaces, sufficiently complete results on the coincidence points existence for set-valued mappings were obtained in [1–3]. These results are based on the covering mapping concept (see [4–9]).

In [10] it was shown that the concept of covering can be introduced for single-valued mappings acting in partially ordered spaces X and Y. Using this concept, the authors obtained theorems on the existence of

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coincidence points for isotone and orderly covering mappings. The present paper is a natural continuation of the results from [10], where the coincidence points problem was considered for single-valued mappings. Here we introduce the concept of covering for set-valued mappings, obtain coincidence points theorems and compare our results with the known theorems on this issue. As a special cases of such statements, there can be derived some classical theorems on fixed points of isotone mappings including the result by R.E. Smithson (see [11]).

In the paper, it is also shown that the known theorems on coincidence points of covering and Lipschitz set-valued mappings in metric spaces follow from the obtained here results for mappings in partially ordered spaces. The corresponding reasoning is based on a modification of the construction of a partial order introduced by R. DeMarr (see [12]), Bishop and Phelps (see [13]) for sets $X \times \mathbb{R}_+$, $Y \times \mathbb{R}_+$, where X, Y are metric spaces, $\mathbb{R}_+ = [0, \infty)$. This construction allows to transform a β-Lipschitz mapping $\Phi : X \rightrightarrows Y$ into an isotone mapping $(x, r) \in X \times \mathbb{R}_+ \mapsto \big\{(y, \beta r) : \, y \in \Phi(x), \; r \geq 0\big\}$, and an α-covering mapping $\Psi : X \rightrightarrows Y$ to an orderly covering mapping $(x, r) \in X \times \mathbb{R}_+ \mapsto \big\{(y, \beta r) : \, y \in \Psi(x), \; r \geq 0\big\}$. So, the results proved in the paper can be applied to mappings in metric spaces.

1. Preliminaries

For the sake of completeness we recall some concepts related to partially ordered sets.

Let X be a nonempty set. Recall that a relation $\preceq$ is a **partial order** on X if it is reflexive, i.e., $x \preceq x$ for all $x \in X$, antisymmetric, i.e., $x_1 \preceq x_2$ and $x_2 \preceq x_1$ imply $x_1 = x_2$, and transitive, i.e., $x_1 \preceq x_2$ and $x_2 \preceq x_3$ imply $x_1 \preceq x_3$. The set X with a partial order $\preceq$ is called a **partially ordered set (or poset)** and is denoted by $(X, \preceq)$.

Assume that $(X, \preceq)$ is a partially ordered set. Below we use the following standard notation: $x \prec u$ if $x \preceq u$ and $x \neq u$, $u \succeq x$ if $x \preceq u$.

If either $x_1 \preceq x_2$ or $x_2 \preceq x_1$, then the points x_1, x_2 are called **comparable**. A subset $S \subset X$ is called a **chain** if any two elements of S are comparable. A sequence $\{x_n\} \subset X$ is called **non-increasing** if $x_{n+1} \preceq x_n$ for all n. A point $q \in X$ is called a **lower bound** of a set $Q \subset X$ if $q \preceq x$ for all $x \in Q$. A lower bound $\bar{q} \in X$ of Q is called the **infimum** of Q, and is denoted by $\inf Q$, if $q \preceq \bar{q}$ for any lower bound q of Q.

A point $m \in Q$ is called **a minimal point** in the set Q if there is no point $q \in Q$ such that $q \prec m$.

We will say that a subset $A \subset X$ is **orderly complete in** X, if for any chain $S \subset A$ there exists $\inf S$ and $\inf S \in A$. If X is orderly complete in X, then we will briefly call $(X, \preceq)$ **orderly complete**. It is obvious that X is orderly complete if and only if any chain $S \subset X$ has an infimum.

We will say that a subset $A \subset X$ is **orderly σ-complete in** X, if for any non-increasing sequence $\{x_n\} \subset A$ there exists $\inf\{x_n\}$ and $\inf\{x_n\} \in A$. If X is orderly σ-complete in X, then we will briefly call $(X, \preceq)$ **orderly σ-complete**. It is obvious that $(X, \preceq)$ is orderly σ-complete if and only if any non-increasing sequence $\{x_n\} \subset X$ has an infimum.

These definitions imply that if $(X, \preceq)$ is orderly complete, then it is orderly σ-complete. The converse is not true. The corresponding example was presented in [10].

2. Coincidence points

In this section, the coincidence points of set-valued mappings are studied. The results similar to coincidence points theorems from [10] for single-valued mappings are obtained.

2.1. Basic concepts

Let $(X, \preceq)$, $(Y, \preceq)$ be partially ordered spaces, $\Psi, \Phi : X \rightrightarrows Y$ be set-valued mappings (i.e. the mappings that assign nonempty subsets $\Psi(x) \subset Y$ and $\Phi(x) \subset Y$ to each point $x \in X$).