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Selective bitopological versions of separability

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ABSTRACT

We study selective versions of separability in bitopological spaces by using the notions of θ-closure and θ-density. Additionally, we consider games associated to these selection properties.

1. Introduction

We first recall the two classical selection principles in topological spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then:

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $\langle A_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle B_n : n \in \mathbb{N}\rangle$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

> For each sequence $\langle A_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle b_n : n \in \mathbb{N}\rangle$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

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If $\mathcal{O}$ denotes the collection of all open covers of X, then the selection principle $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ (resp. $\mathsf{S}_1(\mathcal{O},\mathcal{O})$) is called the *Menger property* (*Rothberger property*).

Let $\mathcal{D}$ denote the family of dense subspaces of a topological space X. In [19] the selection principles $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$ and $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ were first introduced and in [3] the spaces satisfying $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$ and $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ are called M-*separable* and R-*separable*, respectively. Also in [7] the selection properties $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D}^{gp})$ and $\mathsf{S}_1(\mathcal{D},\mathcal{D}^{gp})$ were introduced, where $\mathcal{D}^{gp}$ is the family of groupable dense subsets of a space, later on called H-*separability* (in a little bit modified form) and GN-*separability*, respectively. These selection properties have been studied by several authors [1–4,9,17]. Recently, a new version of selective separability in non-regular topological spaces has been introduced in [5] by using the notions of θ-closure and θ-density.

Although several papers on selective separability have been published so far there are very few papers with bitopological spaces and selection principles which are mainly related to function spaces [11,15,16]. The first study of selection principles theory in the bitopological context began with the paper [13] about selective versions of separability in bitopological spaces and continued in [14], where some results on selection principles in the bitopological context related to function spaces and hyperspaces were obtained.

General references for undefined notions regarding to selection principles in topological spaces include [10,12,18,20].

In this paper we introduce a new version of separability by using θ-closure and θ-density in the bitopological context.

Recall that a point $x \in X$ is a θ-*cluster point* of a subset $S \subseteq X$ if $\mathrm{Cl}(U) \cap S \neq \emptyset$ for each open set U containing x. The set of all θ-cluster points of S is called the θ-*closure* of S and is denoted by $\mathrm{Cl}_\theta(S)$ [21]. A subset $S \subseteq X$ is said to be θ-*dense* in X if $\mathrm{Cl}_\theta(S) = X$. If X contains a countable θ-dense subset, then X is said to be θ-*separable*.

We end this introduction with a few words about the relationship between selection principle theory and game theory.

$\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing a finite set $B_n \subset A_n$. The play $(A_1, B_1, \cdots, A_n, B_n, \cdots)$ is won by TWO if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.

$\mathsf{G}_1(\mathcal{A},\mathcal{B})$ denotes a similar game, but in the n-th round ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

It is evident that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ (resp. $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$) then the selection hypothesis $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ (resp. $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$) is true. The converse implication need not be always true.

2. Bitopological M^θ- and R^θ-separability

Throughout this paper (X,τ_1,τ_2), sometime written simply X, will be a bitopological space (shortly bispace), i.e. the set X endowed with two topologies τ_1 and τ_2. For a subset A of X, $\mathrm{Cl}_i(A)$ will denote the closure of A in (X,τ_i), $i = 1, 2$.

We begin with some definitions we will do with.

Definition 2.1. Let (X,τ_1,τ_2) be a bitopological space. A subset A of X is θ-*bidense* (θ-*double dense* or d_θ-*dense*) in X if A is θ-dense in both (X,τ_1) and (X,τ_2).

X is d_θ-separable if there is a countable set A which is θ-bidense in X. We will denote the family of all θ-dense sets in X by $\mathcal{D}_\theta$.

Now let us denote by $\mathcal{D}_1^\theta$ and $\mathcal{D}_2^\theta$ the collection of all θ-dense subsets of (X,τ_1) and (X,τ_2), respectively.

Definition 2.2. Let (X,τ_1,τ_2) be a bitopological space. Then X is: