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On some problem of Borsuk concerning decompositions of shapes into prime factors

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ABSTRACT

In this paper we show that there exists a 3-dimensional continuum with two different decompositions into a direct product of prime factors in the shape category. This gives the positive answer to a question of K. Borsuk from his monograph "Theory of Shape" (1975).

1. Introduction

We suppose the reader to be familiar with the basic notions and facts of shape theory, which can be found in [4,8,19,5].

Recall that the shapes $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are called *factors* of the shape $\mathrm{Sh}(X \times Y) = \mathrm{Sh}(X) \times \mathrm{Sh}(Y)$ (in both pointed and unpointed cases).

The shape $\mathrm{Sh}(X)$ is said to be *prime,* if it is non-trivial and cannot be decomposed into a product of two non-trivial shapes (see, for example, [2,3], [4, Ch. XII, $11], [5]). Similarly, we define *prime* (or *irreducible*) homotopy types.

Some high dimensional examples of compacta (even polyhedra[1]) with two different decompositions into a product of prime factors in the homotopy (or equivalently, shape) category were published in the sixties-seventies by P. Hilton and J. Roitberg [12], A. Sieradski [24], see also L. Charlap [6].

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[1] Here every polyhedron is assumed to be finite and connected.

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In 1972, at the Topological Conference in Tbilisi, K. Borsuk posed the following question [2, p. 141], published also in his monograph "Theory of Shape" [4, Problem (11.4), p. 356], see also [7, Problem (5.7)]:

Problem 1. Does there exist a 3-dimensional compactum X such that $\mathrm{Sh}(X)$ has two different decompositions into prime factors?

In this note we will show that the answer to this question is positive: there exists a 3-dimensional continuum X such that $Sh(X)$ has two different decompositions into prime factors of dimension 1 and 2 (Theorem 1).

At the Topological Conference in Herceg-Novi (1968) Borsuk asked about decompositions of shapes of compacta into prime factors of dimensions less than 3 [1, p. 100], see also [4, Problem (11.6), p. 357].

Problem 2. Does there exist a compactum X such that $\mathrm{Sh}(X)$ has two different decompositions into a finite number of prime factors which are shapes of at most 2-dimensional compacta?

Theorem 1 gives simultaneously the positive answer to Problem 2.

The questions in consideration may be stated in both, pointed and unpointed, versions. The counterexamples are suitable for both cases. (In the sequel, for the simplicity, the basepoints will be omitted.)

2. Main results

Let us begin with some definitions:

Definition. (1) A group homomorphism $f : A \to S^1$, where S^1 is the circle group, is called a *character* of A. (2) The *character group,* $\chi(A)$, of an Abelian topological group A is the group of all continuous characters with the compact-open topology. (3) If A is a locally compact Abelian group, then the character group $\chi(A)$ is also a locally compact Abelian group (cf. [22,21, Ch. 6]).

Remark 1. For a discrete Abelian group A, A is countable if and only if $\chi(A)$ is metrizable (cf. [9, "Pontryagin duality"]).

Definition. (1) A finite set of elements $a_1, \ldots, a_k$ in an Abelian group is said to be *linearly dependent over* Z if there exist integers $n_1, \ldots, n_k$, not all equal to zero, such that $\sum_{i=1}^{k} n_i a_i = 0$. If such numbers do not exist, the set is *linearly independent.* (2) The *rank* of an Abelian group A is defined as the maximal number of elements of A which are linearly independent over Z (cf. [21]). (Every Abelian group that is not a torsion group has maximal linearly independent sets and the cardinality of all maximal linearly independent sets is the same.)

Remark 2. If A is a discrete Abelian topological group, then $\chi(A)$ is a compact Abelian topological group [21, Ch. 6, p. 235] and $\dim \chi(A) = \operatorname{rank} A$ [21, Theorem 47, p. 259].

Definition. An Abelian group is called *indecomposable* (or *irreducible*) if it is nontrivial and cannot be expressed as a direct product $A \oplus B$ of two proper direct factors A and B (compare [13, Definition 3.1, p. 83]).

The examples of compacta with desired properties will be constructed as the character groups of discrete torsion-free Abelian countable groups with suitable properties and by the Pontryagin duality. This method was earlier used, for example, in [23,18].