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The pseudoarc is a co-existentially closed continuum

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Answering a question of P. Bankston, we show that the pseudoarc is a co-existentially closed continuum. We also show that $C(X)$, for X a nondegenerate continuum, can never have quantifier elimination, answering a question of the first and third named authors and Farah and Kirchberg.

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1. Introduction

A *compactum* is simply a compact Hausdorff space and a *continuum* is a connected compactum. There has been an extensive study of compacta and continua from the model-theoretic perspective; see, for example, [20] or [1]. In [2], Bankston dualizes the model-theoretic notions of existential embeddings and existentially

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closed structures to the categories of compacta and continua; the dual notions are (appropriately named) *co-existential mappings* and *co-existentially closed compacta and continua*. In Appendix A to this note, we show how these notions translate to their usual model-theoretic counterparts in the continuous signature for C*-algebras (e.g., X is a co-existentially closed compactum if and only if $C(X)$ is an existentially closed abelian C*-algebra).

Recall that a continuum X is said to be *indecomposable* if X is not the union of two of its proper subcontinua. If, in addition, every subcontinuum of X is also indecomposable, then X is said to be *hereditarily indecomposable*. In [5], Bankston proves that every co-existentially closed continuum is hereditarily indecomposable.

Amongst the hereditarily indecomposable continua, there is one such continuum that plays a special role. Recall that a continuum X is said to be *chainable* if, for every finite open cover $U_1, \ldots, U_n$ of X, there is a refinement to a cover $V_1, \ldots, V_m$ such that $V_i \cap V_j = \emptyset$ if and only if $|i-j| > 1$. Up to homeomorphism, there is a unique continuum that is both chainable and hereditarily indecomposable; this continuum is called the *pseudoarc* (see [16]). In [4], Bankston asks the natural question: is the pseudoarc a co-existentially closed continuum? In this note, we answer Bankston's question in the affirmative.

The plan of the proof is as follows: motivated by an $\mathcal{L}_{\omega_1,\omega}$ characterization of chainable continua in the (discrete) signature of lattice bases for continua given by Bankston in [6], we prove that the class of separable $C(X)$ for which X is a chainable continuum is definable by a uniform sequence of universal types (in the terminology of [10]) in the continuous signature for C*-algebras. Together with the fact, proven by K.P. Hart in [15], that there is a unique universal theory of $C(X)$ for X a nondegenerate continuum, this allows us to apply the technique of model-theoretic forcing to obtain a metrizable continuum X for which $C(X)$ is existentially closed (so X is hereditarily indecomposable) and for which X is chainable, whence X must be the pseudoarc.

We end this note by observing that $C(X)$, for X a nondegenerate continuum, can never have quantifier elimination. The proof relies on combining the aforementioned result of Hart together with the observation that the class of $C(X)$, for X a continuum, does not have the amalgamation property, as pointed out to us by Logan Hoehn. Together with results from [7], the question of which abelian C*-algebras have quantifier elimination is now completely settled: the only abelian C*-algebras with quantifier elimination are $\mathbb{C}$, $\mathbb{C}^2$, and $C(2^{\mathbb{N}})$. In [7] it was shown that the only non-abelian C*-algebra with quantifier elimination in $M_2(\mathbb{C})$, so we now have the complete list of C*-algebras with quantifier elimination.

In this paper, we assume the reader is familiar with basic model theory as it applies to C*-algebras. A good reference for the unacquainted reader is [11]. For background on the notions appearing in Section 3 (e.g. model companions, model-completions, etc.) we refer the reader to [19].

1.1. Facts from the model theory of continua

In this paper, all C*-algebras are unital and we always work in the (continuous) signature $\mathcal{L}$ for (unital) C*-algebras. In this language, the class of abelian C*-algebras is clearly universally axiomatizable.

Throughout this paper, we will apply a topological adjective to an abelian C*-algebra if its Gelfand spectrum possesses that property. Thus, for example, we will call $C(X)$ connected if X is connected (and thus a continuum).

Fact 1.1. *The class of connected abelian C*-algebras is universally axiomatizable.*

Proof. That the class of connected abelian C*-algebras is closed under ultraproducts is a special case of a more general result due to Gurevic (see [14, Lemma 10]). (Alternatively, if X is compact, then X is connected if and only if $C(X)$ is projectionless; it remains to observe that if $(A_i \mid i \in I)$ is a family of C*-algebras, $\mathcal{U}$ is an ultrafilter on I, and $A = \prod_{\mathcal{U}} A_i$, then any projection of A can be written as $\pi(p_i)$ with

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