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Manifold boundaries

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A B S T R A C T

Given a manifold M, we consider properties of the frontier of a dense metrisable submanifold N of M. We show that if each boundary component of a manifold-with-boundary is compact and the interior is metrisable, then there are only countably many boundary components and the manifold itself is metrisable. We also show how to construct manifolds by adding to a metrisable manifold continuum many sets, such as graphs whose vertices are trees.

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1. Introduction

Nyikos [5, Definition 2.10] defines a space X to be of *Type I* if it is the union of an ω_1-sequence $\{U_\alpha : \alpha < \omega_1\}$ of open subspaces such that $\overline{U_\alpha} \subseteq U_\beta$ whenever $\alpha < \beta$, and such that $\overline{U_\alpha}$ is Lindelöf for all α. Of course any Lindelöf space X is of Type I, simply let $U_\alpha = X$ for each $\alpha < \omega_1$. Each Type I non-metrisable manifold M is associated with an ω_1 tree $\Upsilon(M)$. The tree essentially captures the "shape" of the manifold.

If a manifold M is not of Type I, then it contains a metrisable submanifold N such that the frontier of N is not metrisable. A natural question that arises is: What is the nature of a set that is contained in a manifold and is the frontier of a metrisable submanifold?

The most well known example of a nonmetrisable manifold that is not of Type I is the Prüfer 2-manifold, see [5, Example 3.7] for example. The manifold-with-boundary [5, Example 3.6] (see also a brief description

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in Section 2) from which Nyikos derives the Prüfer manifold has metrisable interior but its boundary contains $\mathfrak{c}$ many ($\mathfrak{c}$ the cardinality of the continuum) copies of $\mathbb{R}$. These copies are discrete, so the manifold contains an uncountable closed discrete set.

We explore the properties of sets that "bound" metrisable manifolds. We consider the components of frontiers of metrisable submanifolds, and we look at the boundary of a manifold-with-boundary where int (M) is metrisable, which may be viewed as a special case of the former. We give bounds on the number of boundary/frontier components, then focus on the structure of these components for 2-manifolds; specifically we consider components that are homeomorphic to trees (where trees are defined as in graph theory). Our goal is to determine what trees can be embedded in a manifold in such a way that the complement of the tree is a metrisable manifold.

In Section 2 we introduce our terminology, especially in relation to trees and what we call their associated wire spaces: generalisations of graph theorists' trees, obtained from countable trees by joining adjacent vertices by an interval of the reals.

In Section 3 we consider compact components. We prove in Theorem 6 a stronger version of:

Theorem 1. *If M is a manifold-with-boundary and int (M) is metrisable, then the number of compact boundary components is countable.*

Thus:

Corollary 2. *If M is a manifold-with-boundary in which each boundary component is compact, and the interior of M is metrisable then M is metrisable.*

In Section 4 we discuss manifolds with boundary and show in Example 8 that it is possible for such a manifold to have $\mathfrak{c}$ many nonmetrisable boundary components, making use of Nyikos's construction of a surface with interior an open half-plane and boundary the long line $\mathbb{L}$.

In Section 5 we discuss the components of the frontier of a dense metrisable submanifold and show in Proposition 10 how to construct a pair of manifolds so that one is metrisable and the other is obtained from it by adding $\mathfrak{c}$ many connected subsets of the unit ball in $\mathbb{R}^n$ whose closure is the set together with a single point of the boundary of the ball.

We characterise in Theorem 17 of Section 6 those wire spaces that may be embedded as closed subspaces of a manifold whose complements are metrisable. This leads to the construction in Corollary 18 of a surface containing a connected, metrisable, dense, open subsurface such that the difference is the disjoint union of all such wire spaces.

2. Preliminaries

In this section we give definitions that will be required in the sequel. Definitions not given here may be found in standard topology text books such as [2]. These include definitions relating to ordinals and in particular an ordinal is the set of its predecessors, so for example $\omega = \{0, 1, \ldots\}$. The cardinality of the continuum is denoted by $\mathfrak{c}$.

The space $\mathbb{R}^n$ is Euclidean space with the usual topology while $\mathbb{B}^n \subset \mathbb{R}^n$ is the closed unit ball. We make use of the ordinals, at least up to $\mathfrak{c}$. We will also think of ω, the first infinite ordinal, as the set of non-negative integers.

A *manifold* is a connected, Hausdorff space M for which there is an integer n such that for each $x \in M$, x has a neighbourhood homeomorphic to $\mathbb{R}^n$. A *manifold-with-boundary* is a connected, Hausdorff space M for which there is an integer n such that each $x \in M$ has a neighbourhood homeomorphic to the closed half-space $\{\langle x_1, \ldots, x_n\rangle \in \mathbb{R}^n : x_n \geq 0\}$. We will assume that our manifolds(-with-boundary) are non-empty. When the dimension n is 2 we will also use the term *surface*.

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