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# Overlays and group actions

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Overlays were introduced by R.H. Fox [7] as a subclass of covering maps. We offer a different view of overlays: it resembles the definition of paracompact spaces via star refinements of open covers. One introduces covering structures for covering maps and $p : X \to Y$ is an overlay if it has a covering structure that has a star refinement. We prove two characterizations of overlays: the first one using existence and uniqueness of lifts of discrete chains, the second one as maps inducing simplicial coverings of nerves of certain covers. We use those characterizations to improve results of Eda–Matijević concerning topological group structures on domains of overlays whose range is a compact topological group.

In case of surjective maps $p : X \to Y$ between connected metrizable spaces, we characterize overlays as local isometries: $p : X \to Y$ is an overlay if and only if one can metrize $X$ and $Y$ in such a way that $p|B(x, 1) : B(x, 1) \to B(p(x), 1)$ is an isometry for each $x \in X$.



## 1. Introduction

R.H. Fox [7] introduced **overlays** $p : X \to Y$ as a subclass of covering maps. Thus, $Y$ has an open cover $\mathcal{U}$ with the property that each $U \in \mathcal{U}$ is evenly covered, i.e. for each $U \in \mathcal{U}$ there is a set $S$ so that each $p^{-1}(U)$, $U \in \mathcal{U}$, can be decomposed as a disjoint union $\bigcup\limits_{s\in S} U_s$ with $p|U_s : U_s \to U$ being a homeomorphism. Such a decomposition will be called a **trivialization** of $p|p^{-1}(U) : p^{-1}(U) \to U$. For $p$ to be an **overlay**, we require that the same indexing set $S$ is chosen for all $U \in \mathcal{U}$. Moreover, if $U, V \in \mathcal{U}$ intersect, then there is a reindexing of elements of decompositions of preimages $p^{-1}(U)$, $p^{-1}(V)$ so that $U_s \cap V_t \neq \emptyset$ implies $s = t$. That allows for extending of trivializations over the union of two elements of the cover $\mathcal{U}$ of $Y$.

**Remark 1.1.** The definitions of overlays in [13] (Definition 1.1) and [12] (the text prior to Proposition 7.2) must be read in the spirit of the above definition. Namely, it is not meant that each $p^{-1}(U)$, $U \in \mathcal{U}$,

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has a fixed decomposition as a disjoint union $\bigcup_{s\in S} U_s$ with $p|U_s : U_s \to U$ being a homeomorphism. The decomposition is fixed in terms of sets, not in terms of indexing by elements of $S$.

The reason overlays are needed is that for general topological spaces one cannot build a theory of covering maps similarly to that for locally connected spaces (see examples in [9] and [4]). Also, see [2] for results comparing classical covering maps and overlays.

In this paper we offer a different view of overlays: it resembles the definition of paracompact spaces via star refinements of open covers. One introduces covering structures for covering maps similarly to overlay structures as in [13]. $p : X \to Y$ is an overlay if it has a covering structure that has a star refinement. This point of view explains why overlays yield a theory superior to covering maps: it is quite analogous to paracompact spaces yielding a much better theory than general topological spaces.

We prove two characterizations of overlays:

1. $p : X \to Y$ is an overlay if and only if there is an open cover $\mathcal{S}$ of $X$ such that every $\mathcal{U}$-chain, $\mathcal{U} = p(\mathcal{S})$, has a lift that is an $\mathcal{S}$-chain and that lift is unique (see Theorem 6.3).

2. $p : X \to Y$ is an overlay if and only if there is an open cover $\mathcal{S}$ of $X$ such that for $\mathcal{U} = p(\mathcal{S})$ the induced map $\mathcal{N}(\mathcal{S}) \to \mathcal{N}(\mathcal{U})$ of nerves of covers is a simplicial covering (see Theorem 8.1 and Corollary 8.2).

Characterization 1 uses ideas of Berestovskii–Plaut [1], later expanded in [3].

In case of surjective maps $p : X \to Y$ between connected metrizable spaces, we characterize overlays as local isometries: $p : X \to Y$ is an overlay if and only if one can metrize $X$ and $Y$ in such a way that $p|B(x, 1) : B(x, 1) \to B(p(x), 1)$ is an isometry for each $x \in X$ (see Theorem 6.4).

One of the main applications of results of the paper is Corollary 8.4 describing overlays over compact topological groups which improves a theorem from [6].

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## 2. A covering map that is not an overlay

In this section we give a simple proof that the covering map of Fox [8] is not an overlay. The usual proofs of that fact are much more complicated (see [8] and [2]).

We begin with a rather easy observation:

**Lemma 2.1.** *If $p : E \to B$ is an overlay, $B$ is a metric space containing an arc $A$, then there is a neighborhood $U$ of $A$ in $B$ such that the projection $p|p^{-1}(U) : p^{-1}(U) \to U$ is a trivial bundle.*

**Proof.** Without loss of generality assume $A = [0, 1]$ and pick a finite open family $\{U_i\}_{i=1}^n$ covering $A$ in $B$ possessing the following properties:

1. $0 \in U_1$, $1 \in U_n$,
2. $U_i$ intersects $U_j$ if and only if $|i - j| \le 1$,
3. each point in $U := \bigcup_{i=1}^n U_i$ belongs to at most 2 elements of the family $\{U_i\}_{i=1}^n$,
4. each $p|p^{-1}(U_i) : p^{-1}(U_i) \to U_i$ is a trivial bundle with a fixed trivialization,
5. any two of the above trivializations can be matched.

Starting from the trivialization of $p|p^{-1}(U_1) : p^{-1}(U_1) \to U_1$ we can extend it by induction to a trivialization of $p|p^{-1}(U) : p^{-1}(U) \to U$. □

We will change slightly the example of Fox [8] using ideas from [4]. Recall the basic construction from [4]: