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Two weak forms of countability axioms in free topological groups ☆

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ABSTRACT

Given a Tychonoff space X, let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group over X in the sense of Markov. For every $n \in \mathbb{N}$, let $F_n(X)$ (resp. $A_n(X)$) denote the subspace of $F(X)$ (resp. $A(X)$) that consists of words of reduced length at most n with respect to the free basis X. In this paper, we discuss two weak forms of countability axioms in $F(X)$ or $A(X)$, namely the csf-countability and snf-countability. We provide some characterizations of the csf-countability and snf-countability of $F(X)$ and $A(X)$ for various classes of spaces X. In addition, we also study the csf-countability and snf-countability of $F_n(X)$ or $A_n(X)$, for $n = 2, 3, 4$. Some results of Arhangel'skiĭ in [1] and Yamada in [20] are generalized. An affirmative answer to an open question posed by Li et al. in [11] is provided.

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1. Introduction

In 1941, Markov [14] introduced the concepts of the free topological group $F(X)$ and the free Abelian topological group $A(X)$ over a Tychonoff space X, respectively. Since then, free topological groups have been a source of various examples and also an interesting topic of study in the theory of topological groups, see [3]. From the algebraic point of view, the structure of $F(X)$ or $A(X)$ is very simple – it is the free algebraic group over the set X. However, the topological structure of $F(X)$ and $A(X)$ is rather complicated even for simple spaces X. For example, it is a well known fact that if X is a non-discrete space, then neither $F(X)$ nor $A(X)$ is Fréchet–Urysohn, and hence first countable, see [1]. This fact motivates researchers to investigate free topological groups in two directions. The first direction of the research on free topological groups is to study some weak forms of countability axioms in $F(X)$ and $A(X)$ over certain classes of spaces X. In this line, Arhangel'skiĭ et al. [4] considered the following questions on $F(X)$ and $A(X)$ over a metrizable space X: *For which spaces X, is $F(X)$ or $A(X)$ a k-space? When is the tightness of $F(X)$ or $A(X)$ countable?* They proved that $F(X)$ is a k-space iff X is locally compact separable or discrete; $A(X)$ is a k-space iff X is locally compact and X' is separable, where X' is the derived set of X. Furthermore, the tightness of $F(X)$ is countable iff X is separable or discrete, and the tightness of $A(X)$ is countable iff X' is separable.

The other direction of research on free topological groups is to study (weak) countability axioms of $F_n(X)$ or $A_n(X)$, where $F_n(X)$ (resp. $A_n(X)$) stands for the subset of $F(X)$ (resp. $A(X)$) formed by all words whose reduced length is at most n. Indeed, Yamada [21] showed that for a metrizable space X, $F_3(X)$ or $A_3(X)$ is Fréchet–Urysohn iff X' is compact, and $F_5(X)$ is Fréchet–Urysohn iff X is compact or discrete. As applications, characterizations of a metrizable space X are given such that $A_n(X)$ is Fréchet–Urysohn for each $n \geq 3$, and $F_n(X)$ is Fréchet–Urysohn for each $n \geq 3$ except for $n = 4$. The subspaces $F_4(X)$ and $A_4(X)$ are very special cases. In [20], Yamada proved that for a metrizable space X, the following are equivalent: (i) $F_n(X)$ is metrizable for each $n \in \mathbb{N}$; (ii) $F_n(X)$ is first countable for each $n \in \mathbb{N}$; (iii) $F_4(X)$ is metrizable; (iv) $F_4(X)$ is first countable; (v) X is compact or discrete. In the same paper, Yamada also studied the first countability of $F_n(X)$ and $A_n(X)$ for $n = 2, 3$. It is proved that for a metrizable space X, the following are equivalent: (i) $F_3(X)$ is metrizable; (ii) $F_3(X)$ is first-countable; (iii) $F_2(X)$ is metrizable; (iv) $F_2(X)$ is first-countable; (v) X' is compact. Furthermore, for a metrizable space X, the following are also equivalent: (i) $A_2(X)$ is first-countable; (ii) $A_2(X)$ is metrizable; (iii) $A_n(X)$ is first-countable for each $n \in \mathbb{N}$; (iv) $A_n(X)$ is metrizable for each $n \in \mathbb{N}$; (v) X' is compact.

Recently, Li et al. [11] continued the study of $F(X)$ and $A(X)$ along the afore-mentioned first direction. They studied several weak forms of countability axioms of $F(X)$ and $A(X)$ defined by networks over some classes of generalized metric spaces X. More precisely, they studied the concepts of sn-networks, cs-networks, cs^*-networks in $F(X)$, $A(X)$, and their subspaces $F_n(X)$ and $A_n(X)$. Two types of countability axioms defined by these concepts, namely snf-countability and csf-countability, were considered. Among many other things, Li et al. established the following results: For a metrizable and crowded space X, $F(X)$ or $A(X)$ is csf-countable iff X is separable; For a stratifiable k-space X, $F(X)$ is snf-countable iff X is discrete. However, the authors of [11] did not consider the snf-countability and csf-countability of $F_n(X)$ and $A_n(X)$.

In the paper, we continue the study of free topological group $F(X)$ and the free Abelian topological group $A(X)$ in the afore-mentioned two directions. In particular, we investigate the csf-countability and the snf-countability of $F(X)$, $A(X)$, $F_n(X)$ and $A_n(X)$ over various classes of generalized metric spaces X. In Section 2, we introduce the necessary notation and terminologies which are used for the rest of the paper. In Section 3, we investigate the snf-countability of free (Abelian) topological groups. First, we provide some characterizations of the snf-countability of $F(X)$, $A(X)$, $F_n(X)$ and $A_n(X)$ over certain classes of topological spaces. The main theorem in this section generalizes a result of Yamada in [20]. Section 4 is devoted to the study of the csf-countability of $F(X)$, $A(X)$, $F_n(X)$ and $A_n(X)$. It is shown