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Some conjectures on continuous rational maps into spheres

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ABSTRACT

Recently continuous rational maps between real algebraic varieties have attracted the attention of several researchers. In this paper we continue the investigation of approximation properties of continuous rational maps with values in spheres. We propose a conjecture concerning such maps and show that it follows from certain classical conjectures involving transformation of compact smooth submanifolds of nonsingular real algebraic varieties onto subvarieties. Furthermore, we prove our conjecture in a special case and obtain several related results.

1. Introduction and main results

Recently several authors devoted their papers to the investigation of continuous rational maps between real algebraic varieties, cf. [3,7,8,13,15–20]. Continuing this line of research, we propose Conjecture A(p), whose proof would completely clarify many problems concerning homotopical and approximation properties of continuous rational maps with values in the unit p-sphere. We prove this conjecture in a special case and also obtain some related results. Furthermore, we show that Conjecture A(p) is a consequence of another conjecture, which has nothing to do with continuous rational maps and originates from the celebrated

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paper of Nash [23] and the subsequent developments due to Tognoli [25], Akbulut and King [1], and other mathematicians. All results announced in this section are proved in Section 2.

Throughout the present paper we use the term *real algebraic variety* to mean a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [4]). The class of real algebraic varieties is identical with the class of quasiprojective real varieties, cf. [4, Proposition 3.2.10, Theorem 3.4.4]. Nonsingular varieties are assumed to be of pure dimension. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let X and Y be real algebraic varieties. A map $f\colon X \to Y$ is said to be *continuous rational* if it is continuous on X and there exists a Zariski open and dense subvariety U of X such that the restriction $f|_U\colon U \to Y$ is a regular map. Let $X(f)$ denote the union of all such U. The complement $P(f) = X \setminus X(f)$ of $X(f)$ is the smallest Zariski closed subvariety of X for which the restriction $f|_{X\setminus P(f)}\colon X \setminus P(f) \to Y$ is a regular map. If $f(P(f)) \neq Y$, we say that f is a *nice* map. There exist continuous rational maps that are not nice, cf. [15, Example 2.2 (ii)]. Continuous rational maps have only recently become the object of serious investigation, cf. [3,7,8,13,15–20]. They form a natural intermediate class between regular and continuous maps. Having many desirable features of regular maps they are more flexible.

The space $\mathcal{C}(X,Y)$ of all continuous maps from X into Y will always be endowed with the compact-open topology. There are the following inclusions

$$\mathcal{C}(X,Y) \supseteq \mathcal{R}^0(X,Y) \supseteq \mathcal{R}_0(X,Y) \supseteq \mathcal{R}(X,Y),$$

where $\mathcal{R}^0(X,Y)$ is the set of all continuous rational maps, $\mathcal{R}_0(X,Y)$ consists of the nice maps in $\mathcal{R}^0(X,Y)$, and $\mathcal{R}(X,Y)$ is the set of regular maps. By definition, a continuous map from X into Y can be approximated by continuous rational maps if it belongs to the closure of $\mathcal{R}^0(X,Y)$ in $\mathcal{C}(X,Y)$. Approximation by nice continuous rational maps or regular maps is defined in the analogous way.

Henceforth we assume that the variety X is compact and nonsingular, and concentrate our attention on maps with values in the unit p-sphere

$$\mathbb{S}^p = \{(u_0,\dots,u_p) \in \mathbb{R}^{p+1} \mid u_0^2 + \cdots + u_p^2 = 1\}$$

for $p \geq 1$. Regular maps from X into $\mathbb{S}^p$ have been extensively studied, cf. [4] and the literature cited there. Here we only recall that the closure of $\mathcal{R}(X,\mathbb{S}^p)$ in $\mathcal{C}(X,\mathbb{S}^p)$ can be a much smaller set than the closure of $\mathcal{R}_0(X,\mathbb{S}^p)$, cf. [17, Example 1.8]. If $\dim X \leq p$, then the set $\mathcal{R}_0(X,\mathbb{S}^p)$ is dense in $\mathcal{C}(X,\mathbb{S}^p)$. This assertion holds for $\dim X < p$ since $\mathbb{R}^p$ is biregularly isomorphic to $\mathbb{S}^p$ with one point removed, whereas for $\dim X = p$ it is proved in [17]. Furthermore, $\mathcal{R}_0(\mathbb{S}^n,\mathbb{S}^p)$ is dense in $\mathcal{C}(\mathbb{S}^n,\mathbb{S}^p)$ for all positive integers n and p, cf. [17]. However, if $\dim X > p$, then it can happen that a continuous map from X into $\mathbb{S}^p$ is not homotopic to any continuous rational map, and hence $\mathcal{R}^0(X,\mathbb{S}^p)$ is not dense in $\mathcal{C}(X,\mathbb{S}^p)$, cf. [17, Theorem 2.8]. There are reasons to believe that homotopical and approximation properties of nice continuous rational maps from X into $\mathbb{S}^p$, investigated in [15] and [17], are actually equivalent and fully determined by certain (co)homological conditions. We give a precise formulation of the last statement in Conjecture A(p).

Some preparation is required. Let M be a compact smooth (of class $\mathcal{C}^\infty$) codimension p submanifold of X. If the normal bundle to M in X is oriented, we denote by τ_M^X the Thom class of M in the cohomology group $H^p(X, X\setminus M;\mathbb{Z})$, cf. [22, p. 118]. The image of τ_M^X by the restriction homomorphism $H^p(X, X\setminus M;\mathbb{Z}) \to H^p(X;\mathbb{Z})$, induced by the inclusion map $X \hookrightarrow (X, X\setminus M)$, will be denoted by $[\![M]\!]^X$ and called the cohomology class represented by M. If X is oriented as a smooth manifold, then $[\![M]\!]^X$ is up to sign Poincaré dual to the homology class in $H_*(X;\mathbb{Z})$ represented by M, cf. [22, p. 136]. Similarly, without any orientability