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Spaces $C(X)$ with ordered bases ☆

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ABSTRACT

The concept of Σ-base of neighborhoods of the identity of a topological group G is introduced. If the index set $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ is unbounded and directed (and if additionally each subset of Σ which is bounded in $\mathbb{N}^{\mathbb{N}}$ has a bound at Σ) a base $\{U_\alpha : \alpha \in \Sigma\}$ of neighborhoods of the identity of a topological group G with $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$ is called a Σ-base (a Σ_2-base). The case when $\Sigma = \mathbb{N}^{\mathbb{N}}$ has been noticed for topological vector spaces (under the name of $\mathfrak{G}$-base) at [2]. If X is a separable and metrizable space which is not Polish, the space $C_c(X)$ has a Σ-base but does not admit any $\mathfrak{G}$-base. A topological group which is Fréchet–Urysohn is metrizable iff it has a Σ_2-base of the identity. Under an appropriate ZFC model the space $C_c(\omega_1)$ has a Σ_2-base which is not a $\mathfrak{G}$-base. We also prove that (i) every compact set in a topological group with a Σ_2-base of neighborhoods of the identity is metrizable, (ii) a $C_p(X)$ space has a Σ_2-base iff X is countable, and (iii) if a space $C_c(X)$ has a Σ_2-base then X is a C-Suslin space, hence $C_c(X)$ is angelic.

1. Preliminaries

In what follows $\mathbb{N}$ will design the set of positive integers equipped with the discrete topology. The product space $\mathbb{N}^{\mathbb{N}}$ is supposed to be provided with the pointwise partial order, i.e., such that $\alpha \leq \beta$ whenever $\alpha(i) \leq \beta(i)$ for every $i \in \mathbb{N}$. In the sequel Σ will always design a topological subspace of $\mathbb{N}^{\mathbb{N}}$. We shall say that a subset Δ of $\mathbb{N}^{\mathbb{N}}$ is (pointwise) *bounded* if $\sup\{\alpha(k) : \alpha \in \Delta\} < \infty$ for every $k \in \mathbb{N}$, otherwise will be called *unbounded*. A covering $\{A_\alpha : \alpha \in I\}$ of a topological space X is said to *swallow* the compact sets if for each compact set Q in X there is $\gamma \in I$ such that $Q \subseteq A_\gamma$. If the covering $\{A_\alpha : \alpha \in I\}$ consists of compact sets, we shall speak of a *compact covering*.

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Unless otherwise stated X will be a (Hausdorff) completely regular space and $C(X)$ will denote the linear space of real-valued continuous functions defined on X. We shall write $C_p(X)$ or $C_c(X)$ when endowed $C(X)$ with the pointwise or the compact-open topology, respectively. All topological spaces are supposed to be Hausdorff. Let us recall that a topological space X is called *C-Suslin* if there is a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ and a map $T:\Sigma\to\mathcal{P}(X)$ such that $\bigcup\{T(\alpha):\alpha\in\Sigma\}=X$ and if $\{\alpha_n\}\subseteq\Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$ and $x_n\in T(\alpha_n)$ for every $n\in\mathbb{N}$ then $\{x_n\}$ has a cluster point in X (see [15]). A topological space X is called *web-compact* if there is a map T from a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ into $\mathcal{P}(X)$ such that $\overline{\bigcup\{T(\alpha):\alpha\in\Sigma\}}=X$ and if $\alpha_n\to\alpha$ in Σ and $x_n\in T(\alpha_n)$ for all $n\in\mathbb{N}$ then $\{x_n\}$ has a cluster point in X (see [13, Definition]). Clearly, every C-Suslin space is web-compact. A topological space X is *angelic* if relatively countably compact sets in X are relatively compact and for every relatively compact subset A of X each point of $\overline{A}$ is the limit of a sequence of A (see [9]). A topological space is *strictly angelic* if X is angelic and every compact subset of X is separable.

A completely regular space X is said to be *M-dominated* by a completely regular space M if there is a compact covering $\mathcal{B}$ of X of the form $\mathcal{B}=\{B_K:K\in\mathcal{K}(M)\}$, where $\mathcal{K}(M)$ stands for the family of all compact sets of M, satisfying that $B_K\subseteq B_Q$ whenever $K\subseteq Q$. If in addition $\mathcal{B}$ swallows the compact sets of X, then X is said to be *strongly M-dominated*, see [3].

In this paper we introduce the notion of a Σ-base of neighborhoods of the identity of a topological group (Definition 3 below), which is a family 'smaller' than a $\mathfrak{G}$-base (see [2]). We show that if X is a separable metrizable space which is not a Polish space, then $C_c(X)$ admits a Σ-base of neighborhoods of the origin but not a $\mathfrak{G}$-base (Theorem 7). We also consider a special type of Σ-bases, named Σ_2-bases, that share some important properties with $\mathfrak{G}$-bases (Definition 11). We show that, under appropriate set-theoretical conditions, there exists a Σ_2-base which is not a $\mathfrak{G}$-base (Example 20). We also prove other results stated in Abstract.

2. Σ-bases and distinguishing examples

A topological group G is said to have a $\mathfrak{G}$-base if there is a base $\{U_\alpha:\alpha\in\mathbb{N}^{\mathbb{N}}\}$ of neighborhoods of the identity e in G such that $U_\beta\subseteq U_\alpha$ whenever $\alpha\leq\beta$. Clearly, every metrizable topological group has a $\mathfrak{G}$-base. Conversely, every Fréchet–Urysohn topological group with a $\mathfrak{G}$-base is metrizable, [11, Theorem 1.2].

A space $C_c(X)$ has a $\mathfrak{G}$-base of (absolutely convex) neighborhoods of the origin if and only if X has a covering $\mathcal{A}=\{A_\alpha:\alpha\in\mathbb{N}^{\mathbb{N}}\}$ made up of compact sets such that (i) $A_\alpha\subseteq A_\beta$ whenever $\alpha\leq\beta$, and (ii) $\mathcal{A}$ swallows the compact sets (see [6, Theorem 2]). Combining this fact with Christensen's theorem (see [5, Theorem 3.3] or [8, Theorem 6.4]) one gets the following result which will be used, not always with explicit mention, along the paper.

Proposition 1. *For a metrizable space X the following are equivalent*

(1) *X is a Polish space.*
(2) *$C_c(X)$ has a $\mathfrak{G}$-base of neighborhoods of the origin.*

Since $X=\mathbb{R}^{\mathbb{N}}$ is Polish but not hemicompact, the previous proposition ensures that $C_c(X)$ is a non-metrizable locally convex space with a $\mathfrak{G}$-base.

Following [14], a (Hausdorff) topological group G has the *strong Pytkeev property* if there exists a sequence $\mathcal{D}$ of subsets of G satisfying the property: for each neighborhood U of the unit e and each $A\subseteq G$ with $e\in\overline{A}\setminus A$, there is $D\in\mathcal{D}$ such that $D\subseteq U$ and $D\cap A$ is infinite. In [10, Theorem 5] we showed that any topological group G with the strong Pytkeev property admits a *quasi*-$\mathfrak{G}$-base $\{U_\alpha:\alpha\in\Sigma\}$ of the identity, i.e., an ordered base of neighborhoods of e over some $\Sigma\subseteq\mathbb{N}^{\mathbb{N}}$.