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Notes on remainders of topological spaces in some compactifications ☆

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ABSTRACT

In this paper, we investigate the compactifications of some topological spaces such that their remainders have countable tightness. We also study addition theorems for compacta. The main results are: (1) If bX is a compactification of a first-countable space X with a G_δ-diagonal (or a space X with a point-countable base) and $bX \setminus X$ has countable tightness, then both bX and $bX \setminus X$ have countable fan-tightness; (2) If a non-locally compact paratopological group G has a compactification bG such that the remainder $bG \setminus G$ is the union of a finite family of metrizable subspaces, then G is locally separable and locally metrizable; (3) If a compact Hausdorff space $Z = X \cup Y$, where X is a non-locally compact topological group which is a σ-space and dense in Z, and Y is a semitopological group, then Z is separable and metrizable; (4) If a compact Hausdorff space $Z = X \cup Y$, where X is a non-locally compact paratopological group that has a countable network and is dense in Z, and Y is a semitopological group, then Z is separable and metrizable. Among them (2) and (3) improve the corresponding results given by A.V. Arhangel'skii in [7].

1. Introduction and preliminaries

One of the most interesting questions on the studying of remainders of a space is that, to what extent, a property of a topological space X is related to another property of some or all the remainders of X. A classical result in this direction is the following theorem due to M. Henriksen and J. Isbell [19]:

Theorem 1.1. *A Tychonoff space X is of countable type if and only if the remainder in any (or some) Hausdorff compactification of X is Lindelöf.*

The following three results are due to Arhangel'skii.

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Theorem 1.2. *([5, Theorem 2.1]) If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

Theorem 1.3. *([6, Theorem 4.2]) If a nowhere locally compact metrizable space X has a compactification bX such that the remainder $bX \setminus X$ has locally a G_δ-diagonal, then bX is separable and metrizable.*

Theorem 1.4. *([3, Theorem 5]) If a non-locally compact topological group X has a compactification bX such that the remainder $bX \setminus X$ has a G_δ-diagonal, then bX is separable and metrizable.*

Recently, the relationship between cardinal invariants of remainders of topological groups and properties of the topological groups itself has been studied extensively. For example, A.V. Arhangel'skii studied topological groups with a remainder that has countable π-character (or pseudocharacter) [2]; A.V. Arhangel'skii and J. Van Mill proved that the character of a non-locally compact topological group G doesn't exceed ω_1 provided that G has a first-countable remainder [13]. In this paper we discuss some topological spaces that have remainders with countable tightness. The results we obtained show that countable tightness coincides with fan-tightness for remainders of some topological spaces.

In references [7] and [11], A.V. Arhangel'skii investigated topological groups and rectifiable spaces whose compactifications are the unions of some special subspaces, and established some new results about what remainders make topological groups and rectifiable spaces metrizable. In this paper we study the compactifications of topological spaces in this direction, and some results given in [7] are improved.

Recall that a paratopological group G is a group G with a topology such that the multiplication is jointly continuous. A semitopological group G is a group G with a topology such that the multiplication is separately continuous. Every paratopological group is a semitopological group and each semitopological group is homogeneous. A space X is of countable type if every compact subset P of X is contained in a compact subset $F \subset X$ that has a countable base in X. All metrizable spaces and all p-spaces are of countable type [4]. A space X is said to have a G_δ-diagonal if the diagonal of X is a G_δ-set in X^2. A sequence $\{\xi_n : n \in \omega\}$ of open covers of a space X is called a development of X if for each $x \in X$, the set $\{st(x, \xi_n) : n \in \omega\}$ is a base at x, where $st(x, \xi_n) = \bigcup\{U \in \xi_n : x \in U\}$. A developable space is a space which has a development. A regular space X is said to be a σ-space if X has a σ-discrete (equivalently, σ-locally finite) network.

Throughout this paper, a space always means a Tychonoff topological space. By a remainder of a Tychonoff space X, we mean the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. $\overline{A}^X$ stands for the closure of A in X.

More terminologies and notations appeared in this paper please refer to [16].

2. Tightness of remainders for some topological spaces

Recall that a topological space X has countable tightness if whenever $A \subset X$ and $x \in \overline{A}$, there exists a countable subset $B \subset A$ such that $x \in \overline{B}$. A topological space X has countable fan-tightness (see [9]) if for any countable family $\{A_n : n \in \omega\}$ of subsets of X satisfying $x \in \bigcap_{n\in\omega} \overline{A_n}$, it is possible to select finite sets $K_n \subset A_n$ in such a way that $x \in \overline{\bigcup_{n\in\omega} K_n}$. Clearly a space with countable fan-tightness has countable tightness. The converse is in general false. A typical example is the Fréchet–Urysohn fan S_ω, a space obtained by identifying all non-isolated points of countably infinite disjoint converging sequences, which is a Fréchet–Urysohn space with uncountable fan-tightness. However, the following result shows that countable tightness coincides with countable fan-tightness for remainders of some topological spaces.

Theorem 2.1. *Let X be a first-countable space with a G_δ-diagonal, and bX be a compactification of X such that the remainder $Y = bX \setminus X$ has countable tightness. Then bX has countable fan-tightness. In particular, Y has countable fan-tightness.*