Topology and its Applications 208 (2016) 64–92

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The discriminant invariant of Cantor group actions

Jessica Dyer, Steven Hurder, Olga Lukina *

Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045, United States

ARTICLE INFO

Article history:
Received 28 January 2016
Received in revised form 6 May 2016
Accepted 9 May 2016
Available online 19 May 2016

MSC:
primary 37B45
secondary 20F22, 37B10

Keywords:
Non-abelian group actions on Cantor sets
Group chains
Regularity
Homogeneity
Discriminant group
Heisenberg group
Dihedral group
Odometers
Ellis semigroup

ABSTRACT

In this work, we investigate the dynamical and geometric properties of weak solenoids, as part of the development of a "calculus of group chains" associated to Cantor minimal actions. The study of the properties of group chains was initiated in the works of McCord [23] and Fokkink and Oversteegen [14], to study the problem of determining which weak solenoids are homogeneous continua. We develop an alternative condition for the homogeneity in terms of the Ellis semigroup of the action, then investigate the relationship between non-homogeneity of a weak solenoid and its discriminant invariant, which we introduce in this work. A key part of our study is the construction of new examples that illustrate various subtle properties of group chains that correspond to geometric properties of non-homogeneous weak solenoids.

1. Introduction

Let X be a Cantor set, let G be a finitely generated infinite group, and let $\Phi\colon G \times X \to X$ be an action by homeomorphisms. That is, there is a map $\Phi\colon G \to Homeo(X)$ which associates a homeomorphism $\Phi(g)$ of X to each $g \in G$. We adopt the short-cut notation $g \cdot x = \Phi(g)(x)$ when convenient. An action (X, G, Φ) is *minimal* if for each $x \in X$, its orbit $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$ is dense in X. In this case, we say that (X, G, Φ) is a *Cantor minimal system*.

Cantor minimal systems (X, G, Φ) and (Y, G, Ψ) are *topologically conjugate*, or just *conjugate*, if there exists a homeomorphism $\tau\colon X \to Y$ such that $\tau(\Phi(g)(x)) = \Psi(g)(\tau(x))$. In this paper, we are concerned with

* Corresponding author.
E-mail addresses: jesscdyer@gmail.com (J. Dyer), hurder@uic.edu (S. Hurder), lukina@math.uic.edu (O. Lukina).

http://dx.doi.org/10.1016/j.topol.2016.05.005

the classification of Cantor minimal systems up to topological conjugacy, and focus especially on invariants of the system for the case when G is non-abelian.

A topological space $\mathcal{S}$ is *homogeneous* if for every $x, y \in \mathcal{S}$ there is a homeomorphism $h\colon \mathcal{S} \to \mathcal{S}$ such that $h(x) = y$. One motivation for this work comes from the problem of classifying solenoids up to homeomorphism as in [3,4], and to understand their groups of self-homeomorphisms.

Recall that a *weak solenoid* $\mathcal{S}$ is an inverse limit of an increasing sequence of non-trivial finite-to-one coverings $\pi_\ell\colon M_\ell \to M_0$ of a closed connected manifold M_0. The projection $f_0\colon \mathcal{S} \to M_0$ is a fiber bundle, whose typical fiber $F_0 = f_0^{-1}(x_0)$ is a Cantor set, for $x_0 \in M_0$. The path lifting property for the finite covering maps π_ℓ induces the monodromy action $\Phi\colon G_0 \times F_0 \to F_0$ of the fundamental group $G_0 = \pi_1(M_0, x_0)$ on the inverse limit fiber F_0. Then (F_0, G_0, Φ) is a Cantor minimal system.

A finite covering $\pi_\ell\colon M_\ell \to M_0$ is said to be *regular* if it is defined by a normal subgroup. That is, let $x_0 \in M_0$ be a chosen basepoint, and let $x_\ell \in M_\ell$ be a lift such that $\pi_\ell(x_\ell) = x_0$, then the covering is regular if the image of the induced map on fundamental groups, $(\pi_\ell)_\#\colon \pi_1(M_\ell, x_\ell) \to \pi_1(M_0, x_0)$, is a normal subgroup of $\pi_1(M_0, x_0)$. It is a standard fact (see [22]) that a covering is regular if and only if the group of deck transformations acts transitively on the fibers of the covering.

If each map $\pi_\ell\colon M_\ell \to M_0$ in the definition of a weak solenoid $\mathcal{S}$ is a regular covering, then we say that $f_0\colon \mathcal{S} \to M_0$ is a *regular solenoid*. The fiber F_0 of a regular solenoid is a Cantor group, and so there is a natural *right* action of the Cantor group F_0 on $\mathcal{S}$ which is transitive on fibers, and commutes with the left monodromy action of G_0 on the fibers. McCord used this fact to show in [23] that a regular solenoid $\mathcal{S}$ is homogeneous. Rogers and Tollefson [25] subsequently gave an example of a weak solenoid which is defined by a sequence of covering maps which are not regular coverings, but the inverse limit space $\mathcal{S}$ is still homogeneous. They posed the problem of determining, under which conditions is a weak solenoid $\mathcal{S}$ homogeneous?

On the other hand, weak solenoids need not be homogeneous, and examples of such were first given by Schori in [27], and later by Rogers and Tollefson in [25]. Fokkink and Oversteegen developed in [14] a criterion for a weak solenoid to be homogeneous, stated as Theorem 1.7 below, formulated in terms of the properties of the nested group chain $\mathcal{G} = \{G_i \mid i \geq 0\}$ of subgroups of finite index in the group $G_0 = \pi_1(M_0, x_0)$, where $G_i \subset G_0$ is the image in G_0 of the fundamental group of the covering space M_i.

The later work of Clark, Fokkink and Lukina [2] gave examples of weak solenoids for which the leaves (the path connected components) in the solenoid have different end structures, and thus these solenoids cannot be homogeneous. The methods used in this paper were geometric in nature, and showed that the geometry of the leaves in a weak solenoid are also obstacles to homogeneity.

The examples of Schori, Rogers and Tollefson, and Clark, Fokkink and Lukina, suggest a variety of questions about the structure of weak solenoids. In this work, we develop tools for their study.

We recall two properties of a Cantor minimal system which will be important for the following. First, a Cantor minimal system (X, G, Φ) is *equicontinuous* with respect to a metric d_X on X, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in X$ and $g \in G$ we have

$$d_X(x, y) < \delta \qquad \Longrightarrow \qquad d_X(g \cdot x, g \cdot y) < \varepsilon.$$

The Cantor minimal system (F_0, G_0, Φ) associated to the fiber of a weak solenoid is equicontinuous.

An *automorphism* of (X, G, Φ) is a homeomorphism $h\colon X \to X$ which commutes with the G-action on X. That is, for every $x \in X$ and $g \in G$, $g \cdot h(x) = h(g \cdot x)$. We denote by $Aut(X, G, \Phi)$ the group of automorphisms of the action (X, G, Φ). Note that $Aut(X, G, \Phi)$ is a topological group using the compact-open topology on maps, and is a closed subgroup of $Homeo(X)$. Given a homeomorphism $\tau\colon X \to Y$ which conjugates the actions (X, G, Φ) and (Y, G, Ψ), then τ induces a topological isomorphism $\tau^*\colon Aut(X, G, \Phi) \cong Aut(Y, G, \Psi)$. Thus, the properties of the group $Aut(X, G, \Phi)$ and its action on the space X are topological conjugacy invariants of the Cantor minimal system.