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Smooth structures on a fake real projective space

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We show that the group of smooth homotopy 7-spheres acts freely on the set of smooth manifold structures on a topological manifold M which is homotopy equivalent to the real projective 7-space. We classify, up to diffeomorphism, all closed manifolds homeomorphic to the real projective 7-space. We also show that M has, up to diffeomorphism, exactly 28 distinct differentiable structures with the same underlying PL structure of M and 56 distinct differentiable structures with the same underlying topological structure of M.

1. Introduction

Throughout this paper M^m will be a closed oriented m-manifold and all homeomorphisms and diffeomorphisms are assumed to preserve orientation, unless otherwise stated. Let $\mathbb{R}\mathbf{P}^n$ be real projective n-space. López de Medrano [10] and C.T.C. Wall [17,18] classified, up to PL homeomorphism, all closed PL manifolds homotopy equivalent to $\mathbb{R}\mathbf{P}^n$ when $n > 4$. This was extended to the topological category by Kirby–Siebenmann [9, p. 331]. Four-dimensional surgery [4] extends the homeomorphism classification to dimension 4.

In this paper we study up to diffeomorphism all closed manifolds homeomorphic to $\mathbb{R}\mathbf{P}^7$. Let M be a closed smooth manifold homotopy equivalent to $\mathbb{R}\mathbf{P}^7$. In section 2, we show that if a closed smooth manifold N is PL-homeomorphic to M, then there is a unique homotopy 7-sphere $\Sigma^7 \in \Theta_7$ such that N is diffeomorphic to $M\#\Sigma^7$, where Θ_7 is the group of smooth homotopy spheres defined by M. Kervaire and J. Milnor in [7]. In particular, M has, up to diffeomorphism, exactly 28 distinct differentiable structures with the same underlying PL structure of M.

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In section 3, we show that if a closed smooth manifold N is homeomorphic to M, then there is a unique homotopy 7-sphere $\Sigma^7 \in \Theta_7$ such that N is diffeomorphic to either $M\#\Sigma^7$ or $\widetilde{M}\#\Sigma^7$, where $\widetilde{M}$ represents the non-zero concordance class of PL-structure on M. We also show that the group of smooth homotopy 7-spheres Θ_7 acts freely on the set of smooth manifold structures on a manifold M.

2. Smooth structures with the same underlying PL structure of a fake real projective space

We recall some terminology from [7]:

Definition 2.1. ([7])

(a) A homotopy m-sphere Σ^m is a smooth closed manifold homotopy equivalent to the standard unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$.
(b) A homotopy m-sphere Σ^m is said to be exotic if it is not diffeomorphic to $\mathbb{S}^m$.
(c) Two homotopy m-spheres Σ_1^m and Σ_2^m are said to be equivalent if there exists a diffeomorphism $f : \Sigma_1^m \to \Sigma_2^m$.

The set of equivalence classes of homotopy m-spheres is denoted by Θ_m. The equivalence class of Σ^m is denoted by $[\Sigma^m]$. M. Kervaire and J. Milnor [7] showed that Θ_m forms a finite abelian group with group operation given by connected sum $\#$ except possibly when $m = 4$ and the zero element represented by the equivalence class of $\mathbb{S}^m$.

Definition 2.2. Let M be a closed PL-manifold. Let (N, f) be a pair consisting of a closed PL-manifold N together with a homotopy equivalence $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are equivalent provided there exists a PL homeomorphism $g : N_1 \to N_2$ such that $f_2 \circ g$ is homotopic to f_1. The set of all such equivalence classes is denoted by $\mathcal{S}^{PL}(M)$.

Definition 2.3 *(Cat = Diff* or *PL-structure sets).* Let M be a closed *Cat*-manifold. Let (N, f) be a pair consisting of a closed *Cat*-manifold N together with a homeomorphism $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are concordant provided there exists a *Cat*-isomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to f_1, i.e., there exists a homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F_{|N_1\times 0} = f_1$ and $F_{|N_1\times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}^{Cat}(M)$.

We will denote the class in $\mathcal{C}^{Cat}(M)$ of (N, f) by $[N, f]$. The base point of $\mathcal{C}^{Cat}(M)$ is the equivalence class $[M, Id]$ of $Id : M \to M$.

We will also denote the class in $\mathcal{C}^{Diff}(M)$ of $(M^n\#\Sigma^n, \mathrm{Id})$ by $[M^n\#\Sigma^n]$. (Note that $[M^n\#\mathbb{S}^n]$ is the class of (M^n, Id).)

Definition 2.4. Let M be a closed PL-manifold. Let (N, f) be a pair consisting of a closed smooth manifold N together with a PL-homeomorphism $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are PL-concordant provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is PL-concordant to f_1, i.e., there exists a PL-homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F_{|N_1\times 0} = f_1$ and $F_{|N_1\times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}^{PDiff}(M)$.

Definition 2.5. Let M^m be a closed smooth m-dimensional manifold. The inertia group $I(M) \subset \Theta_m$ is defined as the set of $\Sigma \in \Theta_m$ for which there exists a diffeomorphism $\phi : M \to M\#\Sigma$.

The concordance inertia group $I_c(M)$ is defined as the set of all $\Sigma \in I(M)$ such that $M\#\Sigma$ is concordant to M.

The key to analyzing $\mathcal{C}^{Diff}(M)$ and $\mathcal{C}^{PDiff}(M)$ are the following results.