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On the geography of simply connected nonspin symplectic 4-manifolds with nonnegative signature

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ABSTRACT

In [8,5], the first author and his collaborators constructed the irreducible symplectic 4-manifolds that are homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for each integer $n \geq 25$, and the families of simply connected irreducible nonspin symplectic 4-manifolds with positive signature that are interesting with respect to the symplectic geography problem. In this paper, we improve the main results in [8,5]. In particular, we construct (i) an infinitely many irreducible symplectic and nonsymplectic 4-manifolds that are homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for each integer $n \geq 12$, and (ii) the families of simply connected irreducible nonspin symplectic 4-manifolds that have the smallest Euler characteristics among the all known simply connected 4-manifolds with positive signature and with more than one smooth structure. Our construction uses the complex surfaces of Hirzebruch and Bauer–Catanese on Bogomolov–Miyaoka–Yau line with $c_1^2 = 9\chi_h = 45$, along with the exotic symplectic 4-manifolds constructed in [2,6,4,7,11].

1. Introduction

Let X be a closed simply connected symplectic 4-manifold, and $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of X, respectively. We define the following two invariants associated to X

$$\chi(X) := (e(X) + \sigma(X))/4 \text{ and } c_1^2(X) := 2e(X) + 3\sigma(X)$$

Recall that if X is a complex surface, then $\chi(X)$ is equal to the holomorphic Euler characteristic $\chi_h(X)$ of X, while $c_1^2(X)$ is equal to the square of the first Chern class of X. A fundamental and challenging problem in the theory of complex surfaces (referred as the geography problem) is the characterization of all ordered pairs of integers (a, b) that can be realized as $(\chi_h(X), c_1^2(X))$ for some minimal complex surface X of general type. The geography problem for complex surfaces was originally introduced and studied by

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Persson in [33], and further progress on this problem was made in [27,36,16,34,35]. It seems presently out of reach to determine all such pairs (a, b) that can be realized, even if one considers the simply complex surfaces with negative signature (see discussion in [13], pages 291–293).

Since all simply connected complex surfaces are Kähler, thus symplectic, it is a natural problem to consider a similar problem for symplectic 4-manifolds. The symplectic geography problem was originally introduced by McCarthy–Wolfson in [28], refers to the problem of determining which ordered pairs of nonnegative integers (a, b) are realized as $(\chi(X),\ c_1^2(X))$ for some minimal symplectic 4-manifold X. The geography problem of simply connected minimal symplectic 4-manifolds has been first systematically studied in [21], then studied subsequently in [18,32,31]). It was shown in [21,18,31]) that many pairs $(\chi,\ c_1^2)$ in negative signature region can be realized with nonspin symplectic 4-manifolds, but there were finitely many lattice points with signature $\sigma < 0$ left unrealized. More recently, it was shown in [4] and the subsequent work in [7], that all the lattice points with signature less than 0 can be realized with simply connected minimal symplectic 4-manifolds with odd intersection form. In terms of the symplectic geography problem, the work in [4,7] concluded that there exists an irreducible symplectic 4-manifold and infinitely many irreducible nonsymplectic 4-manifolds with odd intersection form that realize the following coordinates (χ, c_1^2) when $0 \leq c_1^2 < 8\chi$. A similar results for the nonnegative signature case were obtained in [8,5]. We would like to remark that throughout this paper, we consider the geography problem for nonspin symplectic and smooth 4-manifolds. For the spin symplectic and smooth geography problems, we refer the reader to [32,9,10] and references therein.

Our purpose in this article is to construct new nonspin irreducible symplectic and smooth 4-manifolds with nonnegative signature that are interesting with respect to the symplectic and smooth geography problems. More specifically, we construct i) the infinitely many irreducible symplectic and infinitely many nonsymplectic 4-manifolds that all are homeomorphic but not diffemorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for any $n \geq 12$, and ii) the families of simply connected irreducible nonspin symplectic 4-manifolds with positive signature that have the smallest Euler characteristics among the all known simply connected 4-manifolds with positive signature and with more than one smooth structure. The building blocks for our construction are the complex surfaces of Hirzebruch and Bauer–Catanese on Bogomolov–Miyaoka–Yau line with $c_1^2 = 9\chi_h = 45$, obtained as $(\mathbb{Z}/5\mathbb{Z})^2$ covering of $\mathbb{CP}^2$ branched along a complete quadrangle [13,14] (and their generalization in [15]), and the exotic symplectic 4-manifolds constructed by the first author and his collaborators in [2,6,4,7,11], obtained via the combinations of symplectic connected sum and Luttinger surgery operations. We would like to point out that using our recipe and the family of examples in a very recent preprint of Catanese and Dettweiler [15], one can generalize our construction to obtain examples of simply connected irreducible symplectic 4-manifolds with positive signature that are interesting to the symplectic geography problem. This is explained in subsection 5.4.

Let $\mathbb{CP}^2$ denote the complex projective plane, with its standard orientation, and let $\overline{\mathbb{CP}}^2$ denote the underlying smooth 4-manifold $\mathbb{CP}^2$ equipped with the opposite orientation. Our main results are stated as follows

Theorem 1.1. *Let M be $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for any integer $n \geq 12$. Then there exist an infinite family of nonspin irreducible symplectic 4-manifolds and an infinite family of irreducible nonsymplectic 4-manifolds that all are homeomorphic but not diffeomorphic to M.*

The above theorem improves one of the main results in [8] (see page 11) where exotic irreducible smooth structures on $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for $n \geq 25$ were constructed. Our next theorem improves the main results of [8,5] for the positive signature case (see also the subsection 5.4, where we dealt with the cases of signature greater than 3).

Theorem 1.2. *Let M be one of the following 4-manifolds.*