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Symmetric products of generalized metric spaces

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Dedicated to Professor Sam B. Nadler, Jr., on the occasion of his 75th birthday[3]

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ABSTRACT

We consider several generalized metric properties and study the relation between a space X satisfying such property and its n-fold symmetric product satisfying the same property.

1. Introduction

The hyperspace $CL(X)$ of closed subsets of a topological space equipped with various topologies and various of its subsets such as 2^X, the space of compact subsets of X, and $\mathcal{F}(X)$, the space of finite subsets of X have been the focus of much research. For example, Mizokami presents a survey of results relating a

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generalized metric property of space X with the hyperspaces 2^X and $\mathcal{F}(X)$ [23]. Fisher, Gratside, Mizokami and Shimane prove that for a space X, $CL(X)$ is monotonically normal if and only if X is metrizable, 2^X is monotonically normal if and only if $2^X = \mathcal{F}(X)$ or 2^X is stratifiable. They also show that monotone normality of X^2 is equivalent to the monotone normality of X^n and $\mathcal{F}(X)$ [6] (compare with Theorem 4.6). A survey of $CL(X)$, 2^X and $\mathcal{F}(X)$ with several topologies is in [10]. A study of 2^X and $\mathcal{C}_n(X)$ when X is a compact, connected and metric space can be found in [26] and [17], respectively.

The symmetric products of a space have been less well studied except for the case of symmetric products of continua (compact, connected metric spaces). The n-fold symmetric product $\mathcal{F}_n(X)$ of a space, originally defined in 1931 by Borsuk and Ulam [2], is the quotient of X^n formed by the quotient map $(x_1, x_2, \dots, x_n) \mapsto \{x_1, x_2, \dots, x_n\}$. If X is a Hausdorff space, then $\mathcal{F}_n(X)$ is a closed subset of $CL(X)$ and the union of all symmetric products of X is the subspace $\mathcal{F}(X)$, which is dense in $CL(X)$. Borsuk and Ulam studied the symmetric products of the unit interval $[0,1]$ and showed that $\mathcal{F}_n([0,1])$ is homeomorphic to $[0,1]^n$ for $n \in \{1,2,3\}$, that $\mathcal{F}_n([0,1])$ is not embeddable in the Euclidean space $\mathbb{R}^n$ for any $n \geq 4$, and that $\dim(\mathcal{F}_n([0,1])) = n$ for each n [2]. Borsuk claimed that the third symmetric product of the unit circle $\mathcal{S}^1$ was homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$, where $\mathcal{S}^2$ is the two sphere [1], but Bott showed that actually $\mathcal{F}_3(\mathcal{S}^1)$ is homeomorphic to the three sphere $\mathcal{S}^3$ [3]. Ganea proved that if X is a separable metric space, then $\dim(X^n) = \dim(\mathcal{F}_n(X))$ [7]. Molski showed that $\mathcal{F}_2([0,1]^2)$ is homeomorphic to $[0,1]^4$, that $\mathcal{F}_n([0,1]^2)$ cannot be embedded in $\mathbb{R}^{2n}$ and that $\mathcal{F}_2([0,1]^n)$ cannot be embedded in $\mathbb{R}^{2n}$, for any $n \geq 3$ [25]. Schori characterized $\mathcal{F}_n([0,1])$ as $Cone(D^{n-2}) \times [0,1]$ for some subspace D^{n-2} of $\mathcal{F}_n([0,1])$ [29]. Macías proved that if X is a continuum, then for each $n \geq 3$, each map from $\mathcal{F}_n(X)$ into the unit circle, $\mathcal{S}^1$, is homotopic to a constant map. In particular we have that $\mathcal{F}_n(X)$ is unicoherent for each $n \geq 3$ [15]. He showed that for a finite dimensional continuum X, $\mathcal{C}_1(X)$ is homeomorphic to $\mathcal{F}_2(X)$ if and only if X is homeomorphic to $[0,1]$ [15]; also, $\mathcal{C}_n(X)$ is never homeomorphic to $\mathcal{F}_n(X)$ [18]. Additionally, he proved that if $\mathcal{F}_n(X)$ is a retract of $\mathcal{C}_m(X)$ $(m \geq n)$, then $\mathcal{F}_n(X)$ is uniformly pathwise connected, weakly chainable, movable and has trivial shape [19]. He also obtained some aposyndetic properties of symmetric products of continua [16].

In this paper we study symmetric products of generalized metric spaces. It turns out that the behaviour of the symmetric product topology mirrors the behaviour of the usual product topology. (Where ever possible we have proved our results directly rather than relying on preservation under products and closed maps.) Regarding positive results, in all but two cases (Questions 3.6 and 3.35), we show that $\mathcal{F}_n(X)$ has the generalized metric property if and only if X does. With respect to counterexamples, we find that protometrizability, being a Fréchet space, monotone normality, countable compactness and pseudocompactness do not hold and we give examples of spaces X satisfying each of these properties such that $\mathcal{F}_2(X)$ does not satisfy them. The set-theoretic behaviour of $\mathcal{F}_n(X)$ for a *ccc* space X again mirrors that of X^n and $\mathcal{F}_2(X)$ is *ccc* if and only if X^2 is *ccc*.

We introduce the definitions just before we use them for the first time.

2. Preliminaries

All of our spaces are Hausdorff unless otherwise indicated. The symbol $\mathbb{N}$ stands for the set of positive integers and $\mathbb{R}$ stands for the set of real numbers. If X is a space, 1_X denotes the identity map defined on X. If $f\colon X \to Y$ is a function, $Im(f) = \{f(x) \mid x \in X\}$.

Given a space X, we define its *hyperspaces* as the following sets:

- $CL(X) = \{A \subset X \mid A \text{ is closed and nonempty}\}$;
- $2^X = \{A \in CL(X) \mid A \text{ is compact}\}$,
- $\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},\ n \in \mathbb{N}$;
- $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}$;
- $\mathcal{F}(X) = \{A \in 2^X \mid A \text{ is finite}\}$.