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(H, G)-coincidence theorems for free G-spaces

Francielle R. de C. Coelho [a,1], Edivaldo L. dos Santos [b,*,2]

[a] *Universidade Federal de Uberlândia, Faculdade de Matemática, 38408-100, Uberlândia MG, Brazil*
[b] *Universidade Federal de São Carlos, Departamento de Matemática, 13565-905, São Carlos SP, Brazil*

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A B S T R A C T

Let us consider G a group acting freely on a Hausdorff paracompact topological space X and let Y be a k-dimensional metrizable space (or k-dimensional CW-complex). In this paper, by using the genus of X, $\text{gen}(X, G)$, we prove (H, G)-coincidence theorems for maps $f : X \to Y$. Such theorems generalize the main theorem proved by Aarts, Fokkink and Vermeer in [1] and the main result proved by dos Santos and Coelho in [11].

1. Introduction

Suppose that X, Y are topological spaces, G is a group acting freely on X and $f : X \to Y$ is a continuous map. If H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx$, $g \in G$, then $hy = ghx$. Following [5–7,9], the concept of G-coincidence is generalized as follows: a point $x \in X$ is said to be a (H, G)-*coincidence point* of f if f sends every orbit of the action of H on the G-orbit of x to a single point. If H is the trivial subgroup, then every point of X is a (H, G)-coincidence. If $H = G$, this is the usual definition of coincidence. If $G = \mathbb{Z}_p$, with p prime, then a nontrivial (H, G)-coincidence point is a G-coincidence point.

Aarts, Fokkink and Vermeer [1, Theorem 1] proved that if $i : X \to X$ is a fixed-point free involution of a normal space X with color number $n + 2$ and k is a natural number then for every k-dimensional cone CW-complex Y and every continuous map $\varphi : X \to Y$ there is a $\mathbb{Z}_2$-coincidence, whenever $n \geq 2k$; and this result is the best possible. Let us observe that for $X = S^n$ the result was obtained independently by Shchepin in [12]. Dos Santos and Coelho [11, Theorem 1.1], by using the genus of X, $\text{gen}(X, \mathbb{Z}_p)$, generalized the Aarts, Fokkink and Vermeer's result for free $\mathbb{Z}_p$-actions, where p is prime.

* Corresponding author.
E-mail addresses: francielle@famat.ufu.br (F.R.C. Coelho), edivaldo@dm.ufscar.br (E.L. dos Santos).
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In this paper, we extended the results proved in [1,11] for free G-actions, where G is a finite group. Specifically, we prove the following result:

Theorem 1.1. *Let G be a finite group which acts freely on a Hausdorff paracompact space X, with $\text{gen}\,(X, G) \geq n + 1$ and let k be a natural number.*

(a) *If $n > |G|k$ and Y is a k-dimensional metrizable space, then every continuous map $f : X \to Y$ has a (H, G)-coincidence point, for some nontrivial subgroup $H \subset G$.*

(b) *If $n = |G|k$ and Y is a k-dimensional cone CW-complex, then every continuous map $f : X \to Y$ has a (H, G)-coincidence point, for some nontrivial subgroup $H \subset G$.*

(c) *If $n < |G|k$ and $gen\,(X, G) = n + 1$, then there exists a k-dimensional cone CW-complex Y and a continuous map $f : X \to Y$ such that f has no G-coincidence points. In particular, if $G = H = \mathbb{Z}_p$, f has no (H, G)-coincidence points.*

Remark 1.2. Theorem 1.1 is a natural generalization of [11, Theorem 1.1]. In the particular case, where $G = H \cong \mathbb{Z}_p$, to detect (H, G)-coincidence points with H nontrivial subgroup of G is equivalent to detect G-coincidence points. Also, by using Theorem 2.3, which states that $\text{gen}(X, G) = n + 1$ is equivalent to $\text{col}(X, G) = n + |G|$, we conclude that Theorem 1.1 generalizes [1, Theorem 1].

In the case that Y is a cone CW-complex, Theorem 1.1 shows that the inequality $n \geq |G|k$ is the best condition for the existence of (H, G)-coincidences. Moreover, for $n = |G|k$, this results can not be extended to a wider class of CW-complex Y of dimension k.

Example 1.3. It is enough to consider [11, Example 1.2], with $G = \mathbb{Z}_p$. Consider $Y = \Delta^{ps+p-2}_{s-1}$ (the $(s-1)$-skeleton of the $(ps + p - 2)$-simplex) and $Y^* = \Pi_{i=1}^{p} Y^i - \Delta$, where $Y^i = Y$, for all i and Δ is the diagonal. We have that $G = \mathbb{Z}_p$ acts freely on Y^* and Y^* is a Hausdorff paracompact space. Moreover, it follows from [16] and [2] that $\text{gen}\,(Y^*, \mathbb{Z}_p) = p(s - 1) + 1$.

Define $\pi : Y^* \to Y$ by $\pi(y_1, \ldots, y_p) = y_1$, for all $(y_1, \ldots, y_p) \in Y^*$ and clearly π has no $\mathbb{Z}_p$-coincidence points. From this, we conclude that Theorem 1.1 does not hold in the case $n = |G|\,k$, when Y is any CW-complex.

2. Preliminaries

Aarts, Brouwer, Fokkink and Vermeer, in [2], defined the genus, $\text{gen}\,(X, G)$, in the sense of Švarc, as follows.

Let G be a finite group which acts freely on a Hausdorff paracompact space X. Let G^* denote $G\backslash\{e\}$. We say that an open subset U of X is a *color* if $U \cap g \cdot U = \emptyset$ for all $g \in G^*$ and we shall say that a cover $\mathcal{U}$ of X by colors is a *coloring*. If (X, G) admits a finite coloring, then the *color number* $\text{col}\,(X, G)$ is the minimal cardinality of a coloring. If U is a color, then the set $G \cdot U = \bigcup_{g \in G} g \cdot U$ is called a *set of the first kind* and $G \cdot U$ is said to be *generated* by the color U. As G is a group, the collection $\{g \cdot U |\ g \in G\}$ is pairwise disjoint. The space X together with the group action is usually called a G-space.

Definition 2.1. Suppose that X is a G-space and let U be a color. The *genus*, $\text{gen}\,(X, G)$, is defined as the minimal cardinality of a covering of X by sets of the first kind.

It follows from the definition that the genus in non-decreasing under equivariant maps.

Proposition 2.2. *Let X and Y be Hausdorff paracompact free G-spaces and let $f : X \to Y$ be a G-equivariant map. Then,* $\text{gen}\,(X, G) \leq \text{gen}\,(Y, G)$.