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On a hyperspace of compact subsets which is homeomorphic to a non-separable Hilbert space

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A R T I C L E I N F O

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A B S T R A C T

Let X be a metrizable space and $\mathrm{Comp}(X)$ be the hyperspace consisting of non-empty compact subsets of X endowed with the Vietoris topology. In this paper, we give a necessary and sufficient condition on X for $\mathrm{Comp}(X)$ being homeomorphic to a non-separable Hilbert space. Moreover, we consider the topological structure of pair $(\mathrm{Comp}(\overline{X}), \mathrm{Fin}(X))$ of hyperspaces of X and its completion $\overline{X}$, where $\mathrm{Fin}(X)$ is the hyperspace of non-empty finite sets in X.

1. Introduction

Throughout this paper, we assume that spaces are metrizable, maps are continuous and κ is an infinite cardinal. Given a space X, let $\mathrm{Comp}(X)$ be the hyperspace of non-empty compact sets in X with the Vietoris topology. In infinite-dimensional topology, it is an important subject to determine topological structures of naturally occurring spaces such as hyperspaces. Especially, the hyperspace $\mathrm{Comp}(X)$ is a classical object and has been studied, see, for instance, [2,3]. D.W. Curtis [2] gave a necessary and sufficient condition on X for $\mathrm{Comp}(X)$ being homeomorphic to the separable Hilbert space as follows:

Theorem 1.1. *For a space X, $\mathrm{Comp}(X)$ is homeomorphic to the separable Hilbert space if and only if X is separable, connected, locally connected, topologically complete and nowhere locally compact.*

We denote the Hilbert space of density κ by $\ell_2(\kappa)$. In the case that κ is uncountable, M. Yaguchi [14, Theorem 1.1] showed the following theorem:

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Theorem 1.2. *If a space X is homeomorphic to $\ell_2(\kappa)$, then so is* $\mathrm{Comp}(X)$.

In this paper, we characterize a space X whose hyperspace $\mathrm{Comp}(X)$ is homeomorphic to a Hilbert space of density κ as follows:

Main Theorem. *Let X be a space. The hyperspace $\mathrm{Comp}(X)$ is homeomorphic to $\ell_2(\kappa)$ if and only if X is connected, locally connected, topologically complete, nowhere locally compact, and each non-empty open set in X is of density κ.*

Let $\mathrm{Fin}(X) \subset \mathrm{Comp}(X)$ be the hyperspace of non-empty finite subsets of a space X. By $\ell_2^f(\kappa)$, we mean the linear subspace spanned by the canonical orthonormal basis of $\ell_2(\kappa)$. D.W. Curtis and N.T. Nhu [5] characterized a space X whose hyperspace $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\omega)$, and the author [7] generalized it as follows:

Theorem 1.3. *For a space X, $\mathrm{Fin}(X)$ is homeomorphic to $\ell_2^f(\kappa)$ if and only if X is connected, locally path-connected, strongly countable-dimensional, σ-locally compact*[1] *and any non-empty open subset of X is of density κ.*

For spaces X and Y, writing (X, Y), we understand Y is a subspace of X. A pair (X, Y) of spaces is homeomorphic to (X', Y') if there exists a homeomorphism $f : X \to X'$ such that $f(Y) = Y'$. A subset A of a space X is *locally non-separating* in X if for every non-empty connected open set U in X, $U \setminus A$ is non-empty and connected. As a corollary of Main Theorem and the paper [7], we can establish the following:

Corollary 1.4. *Let X be a connected, locally path-connected, strongly countable-dimensional and σ-locally compact space such that any non-empty open subset of X is of density κ. Suppose that X has a locally connected and nowhere locally compact completion $\overline{X}$. Then the pair $(\mathrm{Comp}(\overline{X}), \mathrm{Fin}(X))$ is homeomorphic to $(\ell_2(\kappa), \ell_2^f(\kappa))$ if and only if $\overline{X} \setminus X$ is locally non-separating in $\overline{X}$.*

2. Toruńczyk's characterization of Hilbert manifolds

In this section, we introduce Toruńczyk's characterization of Hilbert manifolds that will be used to prove Main Theorem. It is said that a space X has *the countable locally finite approximation property* if for each open cover $\mathcal{U}$ of X, there exists a sequence $\{f_n : X \to X\}_{n<\omega}$ of maps such that every f_n is $\mathcal{U}$-close to the identity map on X and the family $\{f_n(X)\}_{n<\omega}$ of images is locally finite in X. Recall that for maps $f : X \to Y$ and $g : X \to Y$, and for an open cover $\mathcal{U}$ of Y, f is *$\mathcal{U}$-close* to g if for each point $x \in X$, there exists a member $U \in \mathcal{U}$ such that $f(x)$ and $g(x)$ are contained in U. We denote the closed unit interval $[0, 1]$ by $\mathbf{I}$. For $n < \omega$, a space X has *the κ-discrete n-cells property* provided that the following condition holds:

- For every open cover $\mathcal{U}$ of X and every map $f : \bigoplus_{\gamma<\kappa} A_\gamma \to X$, where each $A_\gamma = \mathbf{I}^n$, there is a map $g : \bigoplus_{\gamma<\kappa} A_\gamma \to X$ such that g is $\mathcal{U}$-close to f and $\{g(A_\gamma)\}_{\gamma<\kappa}$ is discrete in X.

This property is one of important ingredients in characterizing non-separable Hilbert manifolds. More details can be found in [1]. H. Toruńczyk [10,11] gave the following celebrated characterization to Hilbert manifolds (cf. [1, Theorem 3.1]):

Theorem 2.1. *A connected space X of density κ is a Hilbert manifold if and only if the following conditions are satisfied:*

[1] A space X is σ-locally compact if it is a countable union of locally compact subsets.