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# The structure of Whitney blocks ☆

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Let $X$ be a metric continuum and $C(X)$ the hyperspace of subcontinua of $X$. A Whitney block is a set of the form $\mu^{-1}([s,t])$, where $\mu : C(X) \to [0,1]$ is a Whitney map and $0 \leq s < t < 1$. In this paper we study continua for which Whitney blocks are homeomorphic to $X \times [0,1]$. We characterize an arc, a simple closed curve and simple $n$-ods in terms of Whitney blocks. We also show that if $X$ is arc-like, then each Whitney block for $C(X)$ is 2-cell-like; and if $X$ is circle-like, then each Whitney block of the form $\mu^{-1}([0,t])$ is ring-like.



## 1. Introduction

A *continuum* is a compact connected metric space with more than one point. Given a continuum $X$, a *subcontinuum* of $X$ is a closed, nonempty and connected subset of $X$. That is, $A \subset X$ is a subcontinuum of $X$ if $A$ is a continuum with the induced topology or $A$ is a one-point set. We consider the hyperspace

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$C(X)$ defined as

$$C(X) = \{A \subset X : A \text{ is a subcontinuum of } X\}.$$

The hyperspace $C(X)$ is considered with the Hausdorff metric $H$ [4, Theorem 2.2].

We also consider the subset of the hyperspace consisting of singletons $F_1(X) = \{\{p\} \in C(X) : p \in X\}$, which is isometric to $X$.

A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0, 1]$ such that:

(a) $\mu(\{p\}) = 0$ for each $p \in X$,

(b) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$ and

(c) $\mu(X) = 1$.

It is known [4, Theorem 13.4] that for every continuum $X$, $C(X)$ admits Whitney maps.

A *Whitney level for* $C(X)$ is a set of the form $\mu^{-1}(t)$, where $0 < t < 1$. A *Whitney block for* $C(X)$ is a set of the form $\mu^{-1}([s, t])$, where $0 \leq s < t < 1$. An *initial Whitney block* is a set of the form $\mu^{-1}([0, t])$, where $0 < t < 1$.

Whitney levels have been widely studied. Many properties and references about Whitney levels can be found in [4]. On the other hand, there are only few known properties for Whitney blocks. In this paper and in [1] we introduce a systematic study on these blocks.

In this paper we study continua $X$ for which Whitney blocks are homeomorphic to $X \times [0, 1]$. We characterize arcs, simple closed curves and simple $n$-ods in terms of their Whitney blocks. We show that if $X$ is arc-like, then each Whitney block for $C(X)$ is 2-cell-like; and if $X$ is circle-like, then each initial Whitney block for $C(X)$ is ring-like.

## 2. Some models for Whitney blocks

An *$n$-cell* is a continuum homeomorphic to the product $[0, 1]^n$. A *ring* is a continuum homeomorphic to $S^1 \times [0, 1]$.

Given an integer $n \geq 3$, a *simple $n$-od* is a continuum $X$ homeomorphic to the cone over a discrete space with $n$ points. The *vertex* of $X$ is the vertex of the cone and the *legs* of the $X$ are the arcs that join the vertex to one point in the base of the cone. A *simple* 3*-od* is a simple triod. A *triod* is a continuum $X$ for which there is a subcontinuum $B$ such that $X - B$ has at least three components.

For a continuum $X$ and $A, B \in C(X)$ such that $A \subset B$, an *order arc from $A$ to $B$* is a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(u) \subset \alpha(v)$ if $0 \leq u \leq v \leq 1$. It is known [4, Theorem 14.6] that if $A, B \in C(X)$ and $A \subset B$, then there exist order arcs from $A$ to $B$.

**Theorem 2.1.** *Whitney blocks for $C([0, 1])$ are* 2*-cells.*

**Proof.** Let $\mu : C([0, 1]) \to [0, 1]$ be a Whitney map, $0 \leq s < t < 1$ and $\mathcal{A} = \mu^{-1}([s, t])$.

Let $\sigma : C([0, 1]) - \{[0, 1]\} \to [0, 1]$ be given by $\sigma([a, b]) = \frac{a}{1-b+a}$.

Then $\sigma$ has the following properties:

(a) $\sigma$ is well defined and continuous,

(b) $\sigma([a, b]) = 0$ if and only if $a = 0$,

(c) $\sigma([a, b]) = 1$ if and only if $b = 1$.

Define $\varphi : \mathcal{A} \to [0, 1]^2$ by

$$\varphi([a, b]) = (\sigma([a, b]), \tfrac{\mu([a,b])-s}{t-s}).$$

Clearly, $\varphi$ is a well defined continuous function.