Topology and its Applications 206 (2016) 228–240

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The rank of a warping matrix

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A R T I C L E I N F O

Article history:
Received 25 August 2015
Received in revised form 15 March 2016
Accepted 5 April 2016
Available online 12 April 2016

MSC:
57M25

Keywords:
Knot diagram
Warping degree
Warping matrix

A B S T R A C T

The warping matrix has been defined for knot projections and knot diagrams by using warping degrees, and the warping matrix of a knot diagram represents the knot diagram uniquely. In this paper we show that the rank of a warping matrix is one greater than the crossing number of the knot projection or diagram. We also discuss the linearly independence of knot diagrams by considering the warping incidence matrices.

1. Introduction

In [8], Kawauchi introduced the notion of warping degrees of oriented knot and link diagrams. In [14] and [15], the properties of the warping degrees of knot and link diagrams were studied. For example, inequalities of the warping degree and the crossing number of a knot and link diagram characterizing alternating diagrams were given there. Warping degrees of spatial graphs, virtual knots and nanowords are also studied (see, for example, [4,7,9,11]). In [17] and [10], the warping polynomial and the warping crossing polynomial of a knot diagram were defined via the warping degree labeling and the warping degree sequence which have the information of warping degrees of the diagram, and studied (see also [5]) and applied to define polynomial invariants for virtual knots (see [1,2,6]). The span of a warping polynomial and a warping degree sequence was also studied in [17] and applied to define an alternating distance in [13]. (Recently, it was shown in [12] that every knot has a diagram such that the span is two.) In [16], the warping matrix of a

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http://dx.doi.org/10.1016/j.topol.2016.04.003
0166-8641/

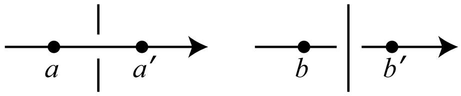

Fig. 1. $d(D_{a'}) = d(D_a) + 1$ and $d(D_{b'}) = d(D_b) - 1$.

knot projection and a knot diagram was defined by the second author, which has the information of warping degrees and warping polynomials, and it is shown that there is a one-to-one correspondence between oriented knot diagrams on S^2 and warping matrices. Hence the warping matrix can be used as a notation for oriented knots. However, it is not easy to calculate warping matrices for knot projections and diagrams with a large crossing number; the size of the warping matrix of a knot projection with c crossings is $2^c \times 2c$, and that of a knot diagram is $(2^c - 1) \times 2c$. One of the motivations on the study of warping matrices is to reduce the size of the matrix. The following theorem would be helpful if we can approach this matter with the theory of matrix decompositions such as the LDU decomposition:

Theorem 1.1. *Let P be an oriented knot projection on S^2, and $M(P)$ the warping matrix of P. We have the following equality:*

$$\operatorname{rank} M(P) = c(P) + 1,$$

where $c(P)$ is the crossing number of P.

Let D be an oriented knot diagram on S^2, and $c(D)$ the crossing number of D. Let $\overline{M}(D)$ be the *warping matrix of D without signs*, which is mentioned concretely in Section 2. We also have the following theorem:

Theorem 1.2. *We have*

$$\operatorname{rank} \overline{M}(D) = c(D) + 1.$$

The rest of this paper is organized as follows: In Section 2, we review the warping matrix. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we discuss warping matrices without signs for the trivial knots and prime alternating knots. In Section 5, we define a new matrix, the warping incidence matrix of a knot diagram. In Section 6, we investigate the linearly independence for knot diagrams.

2. Warping matrix

In this section we review the warping degree and warping matrix. See [14] and [16] for details. In this paper, we assume that all knot projections and diagrams are oriented. Let D be a knot diagram on S^2. Take a base point of D avoiding crossing points. We denote by D_b the pair of D and b. A crossing point p is said to be a *warping crossing point* of D_b if we meet p as an undercrossing first (and an overcrossing later) when we travel D from b (we encounter each crossing twice). The *warping degree* $d(D_b)$, which was defined by Kawauchi, of D_b is defined to be the number of the warping crossing points of D_b. As shown in [14], we have the following:

Proposition 2.1. *(Lemma 2.5 in [14]) When a base point is moved through an overcrossing (resp. undercrossing) on a knot diagram with the orientation, the warping degree increases (resp. decreases) by one.*

That is, we have $d(D_{a'}) = d(D_a) + 1$ and $d(D_{b'}) = d(D_b) - 1$ for a knot diagram D with the base points illustrated in Fig. 1.