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The homotopy type of spaces of coprime polynomials revisited

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A B S T R A C T

The purpose of this paper is to study the topology of certain toric varieties X_I, arising as quotients of the action of $\mathbb{C}^*$ on complements of arrangements of coordinate subspaces in $\mathbb{C}^n$, and to improve the homotopy stability dimension for the inclusion map $i_d : \mathrm{Hol}_d^*(S^2, X_I) \to \mathrm{Map}_d^*(S^2, X_I)$ given in [11] by making use of the Vassiliev spectral sequence.
We also improve the homotopy stability dimension of this inclusion given by G. Segal [18] for $X_I = \mathbb{C}\mathrm{P}^{n-1}$ and $n \geq 3$.

1. Introduction

1.1. Coordinate subspaces and the spaces X_I

Let $n \geq 2$ be a positive integer and let $[n]$ denote the set $[n] = \{0, 1, 2, \cdots, n-1\}$. For each subset $\sigma = \{i_1, \cdots, i_s\} \subset [n]$, let $L_\sigma \subset \mathbb{C}^n$ denote *the coordinate subspace in* $\mathbb{C}^n$ defined by

$$L_\sigma = \{(x_0, x_1, \cdots, x_{n-1}) \in \mathbb{C}^n : x_{i_1} = \cdots = x_{i_s} = 0\}. \tag{1.1}$$

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Let I be any collection of subsets of $[n]$ such that $\mathrm{card}(\sigma) \geq 2$ for all $\sigma \in I$, where $\mathrm{card}(\sigma)$ denotes the number of elements in σ. Let $Y_I \subset \mathbb{C}^n$ be *the complement of the arrangement of coordinate subspaces* defined by

$$Y_I = \mathbb{C}^n \setminus \bigcup_{\sigma \in I} L_\sigma = \mathbb{C}^n \setminus L(I), \quad \text{where we set } \ L(I) = \bigcup_{\sigma \in I} L_\sigma. \tag{1.2}$$

Consider the natural free $\mathbb{C}^*$-action on Y_I given by coordinate-wise multiplication and let X_I denote the orbit space given by

$$X_I = Y_I/\mathbb{C}^* = (\mathbb{C}^n \setminus L(I))/\mathbb{C}^*. \tag{1.3}$$

Note that X_I coincides with the complex variety considered in [11, page 437], and that there is a principal $\mathbb{C}^*$-bundle

$$Y_I \xrightarrow{p_I} X_I. \tag{1.4}$$

Example 1.1. (i) If $I = I(n) = \{\{0, 1, \cdots, n-1\}\}$, $L(I(n)) = \{\mathbf{0}\}$ and we can identify $X_{I(n)}$ with the $(n-1)$-dimensional complex projective space $\mathbb{C}\mathrm{P}^{n-1}$, i.e. $X_{I(n)} = (\mathbb{C}^n \setminus \{\mathbf{0}\})/\mathbb{C}^* = \mathbb{C}\mathrm{P}^{n-1}$.

(ii) If $n \geq 3$ and $I = J(n) = \{\{i, j\} : 0 \leq i < j < n\}$, we can identify $X_{J(n)}$ with the subspace of $\mathbb{C}\mathrm{P}^{n-1}$ given by $X_{J(n)} = \mathbb{C}\mathrm{P}^{n-1} \setminus \bigcup_{0 \leq i < j < n} H_{i,j}$, where $H_{i,j} = \{[x_0 : \cdots : x_{n-1}] \in \mathbb{C}\mathrm{P}^{n-1} : x_i = x_j = 0\}$.[1] In general, we easily see that $X_I = \mathbb{C}\mathrm{P}^{n-1} \setminus \bigcup_{\sigma \in I} H_\sigma$, where

$$H_\sigma = \{[x_0 : \cdots : x_{n-1}] \in \mathbb{C}\mathrm{P}^{n-1} : x_j = 0 \text{ for all } j \in \sigma\}. \qquad \square$$

The algebraic torus $\mathbb{T}^{n-1}_{\mathbb{C}} = (\mathbb{C}^*)^{n-1}$ acts on X_I in the natural manner

$$(t_1, \cdots, t_{n-1}) \cdot [x_0 : \cdots x_{n-1}] = [x_0 : t_1 x_1 : \cdots : t_{n-1} x_{n-1}] \tag{1.5}$$

for $((t_1, \cdots, t_{n-1}), [x_0 : \cdots : x_{n-1}]) \in \mathbb{T}^{n-1}_{\mathbb{C}} \times X_I$, and it is easy to see that X_I is a smooth toric variety. Note that X_I is a non-compact toric variety (its fan is not complete) if $I \neq I(n)$.

1.2. *The simplicial complex $K(I)$*

There is an alternative and better known way to construct the spaces X_I. Recall that *a simplicial complex* K on an index set $[n] = \{0, 1, 2, \cdots, n-1\}$ is a collection of subsets σ of $[n]$ which satisfies the condition that any $\tau \subset \sigma$ is contained in K if $\sigma \in K$.[2] For a simplicial complex K on the index set $[n]$, let $U(K)$ denote *the complement of the arrangement of coordinate subspaces* given by

$$U(K) = \mathbb{C}^n \setminus \bigcup_{\sigma \notin K, \sigma \subset [n]} L_\sigma. \tag{1.6}$$

Now recall the following useful result.

[1] To simplify the notation we will write X_n for $X_{J(n)}$, as in [11].

[2] In this paper a simplicial complex means *an abstract* simplicial complex and we assume that any simplicial complex contains the empty set $\emptyset$.