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Almost meshed locally connected continua have unique second symmetric product

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For a metric continuum X and a positive integer n, we consider the n-th symmetric product $F_n(X)$ of all nonempty subsets of X with at most n points, with the Hausdorff metric. In this paper we prove for an almost meshed locally connected continuum X, that if Y is a continuum and $F_2(X)$ is homeomorphic to $F_2(Y)$, then X is homeomorphic to Y. This answers a question by Alejandro Illanes.

1. Introduction

The study of the uniqueness of hyperspaces of metric continua is an area of extensive research [1–5, 7,8,10–28,30]. Of particular interest is the n-th symmetric product of a metric continuum, whose most recent results establish that wired continua have unique hyperspace $F_n(X)$, for $n \geq 4$ [14], and that meshed continua have unique hyperspace $F_n(X)$, for each $n \in \{2, 3\}$ [11]. In this paper, we prove that almost meshed locally connected continua have unique hyperspace $F_2(X)$.

A *continuum* is a compact connected metric space, with more than one point. The set of positive integers is denoted by $\mathbb{N}$. Given a continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

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$$2^X = \{A \subset X \colon A \text{ is a nonempty closed subset of } X\},$$
$$C_n(X) = \{A \in 2^X \colon A \text{ has at most } n \text{ components}\}, \text{ and}$$
$$F_n(X) = \{A \in 2^X \colon A \text{ has at most } n \text{ points}\}.$$

All these hyperspaces are metrized by the Hausdorff metric [29, Theorem 2.2]. The hyperspaces $F_n(X)$ and $C_n(X)$ are called the *n-th symmetric product of* X and the *n-fold hyperspace* of X, respectively.

For a continuum X, let $\mathcal{H}(X)$ denote any one of the hyperspaces defined above. We say that a continuum X has *unique hyperspace* $\mathcal{H}(X)$ provided that the following implication holds: if Y is a continuum and $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, then X is homeomorphic to Y.

The topic of this paper is inserted in the following general problem.

Problem. Find conditions on the continuum X, so that X has unique hyperspace $\mathcal{H}(X)$.

A *finite graph* is a continuum that can be written as the union of finitely many arcs, each two of which are either disjoint or intersect only in one or both of their end points. Let $\mathfrak{G} = \{X: \ X \text{ is a finite graph}\}$; the following results hold for $\mathfrak{G}$.

(a) If $X \in \mathfrak{G}$ and $n \in \mathbb{N}$, then X has unique hyperspace $F_n(X)$, see [7, Corollary 5.9].

A *dendrite* is a locally connected continuum without simple closed curves. Let $\mathfrak{D} = \{X \colon X$ is a dendrite whose set of end points is closed$\}$. Notice that $\mathfrak{G} \not\subset \mathfrak{D}$ and $\mathfrak{D} \not\subset \mathfrak{G}$. The following result for $\mathfrak{D}$ has been proved.

(b) If $X \in \mathfrak{D}$ and $n \in \mathbb{N}$, then X has unique hyperspace $F_n(X)$, see [2, Theorem 5.2], [18, Theorem 3.7].

Given a continuum X, let

$$\mathcal{G}(X) = \{x \in X \colon x \text{ has a neighborhood } G \text{ in } X \text{ such that } G \text{ is a finite graph}\},$$

and let

$$\mathcal{P}(X) = X - \mathcal{G}(X).$$

Following [12], a continuum X is said to be *almost meshed* provided that the set $\mathcal{G}(X)$ is dense in X; an almost meshed continuum X is *meshed* if it has a basis of neighborhoods $\mathfrak{B}$ such that $U - \mathcal{P}(X)$ is connected, for each element $U \in \mathfrak{B}$. Let $\mathcal{AM} = \{X \colon X \text{ is an almost meshed continuum}\}$, $\mathcal{M} = \{X \colon X \text{ is a meshed continuum}\}$, and $\mathcal{LC} = \{X \colon X \text{ is a locally connected continuum}\}$.

(c) If $X \in \mathcal{AM} \cap \mathcal{LC}$ and $n \in \mathbb{N} - \{2, 3\}$, then X has unique hyperspace $F_n(X)$, [22, Corollary 4.4].
(d) If $X \in \mathcal{M}$ and $n \in \{2, 3\}$, then X has unique hyperspace $F_n(X)$, [11, Theorem 3.11].

We recall that a *wire* in a continuum X is a subset α of X such that α is homeomorphic to one of the spaces $(0, 1)$, $[0, 1)$, $[0, 1]$ or the unit circle S^1 in the Euclidean plane, and α is a component of an open subset of X [13]. Given a continuum X, let $W(X) = \bigcup\{\alpha \subset X \colon \alpha \text{ is a wire in } X\}$. The continuum X is said to be *wired* whenever $W(X)$ is dense in X. Let $\mathcal{W} = \{X \colon X \text{ is a wired continuum}\}$.

(e) If $X \in \mathcal{W}$ and $n \in \mathbb{N} - \{2, 3\}$, then X has unique hyperspace $F_n(X)$, [14, Corollary 6].