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Nonstandard homology theory for uniform spaces

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ABSTRACT

We introduce a new homology theory of uniform spaces, provisionally called μ-homology theory. Our homology theory is based on hyperfinite chains of microsimplices. This idea is due to McCord. We prove that μ-homology theory satisfies the Eilenberg–Steenrod axioms. The characterization of chain-connectedness in terms of μ-homology is provided. We also introduce the notion of S-homotopy, which is weaker than uniform homotopy. We prove that μ-homology theory satisfies the S-homotopy axiom, and that every uniform space can be S-deformation retracted to a dense subset. It follows that for every uniform space X and any dense subset A of X, X and A have the same μ-homology. We briefly discuss the difference and similarity between μ-homology and McCord homology.

1. Introduction

McCord [1] introduced a homology of topological spaces using nonstandard methods. McCord's theory is based on hyperfinite chains of microsimplices. Intuitively, microsimplices are abstract simplices with infinitesimal diameters. Garavaglia [2] proved that McCord homology coincides with Čech homology for compact spaces. Živaljević [3] proved that McCord cohomology also coincides with Čech cohomology for locally contractible paracompact spaces. Korppi [4] proved that McCord homology coincides with Čech homology with compact supports for regular Hausdorff spaces.

In this paper, we introduce a new microsimplicial homology theory of uniform spaces, provisionally called μ-homology theory. μ-homology theory satisfies the Eilenberg–Steenrod axioms. Vanishing of the 0-th reduced μ-homology characterizes chain-connectedness. We also introduce the notion of S-homotopy, which is weaker than uniform homotopy. μ-homology theory satisfies the S-homotopy axiom. Hence μ-homology is an S-homotopy invariant. Every uniform space can be S-deformation retracted to a dense subset. It follows that for every uniform space X and any dense subset A of X, X and A have the same μ-homology. We briefly discuss the difference and similarity between μ-homology and McCord homology.

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The basics of nonstandard analysis are assumed. We fix a universe $\mathbb{U}$, the standard universe, satisfying sufficiently many axioms of ZFC. All standard objects we consider belong to $\mathbb{U}$. We also fix an elementary extension ${}^*\mathbb{U}$ of $\mathbb{U}$, the internal universe, that is $|\mathbb{U}|^+$-saturated. The map $x \mapsto {}^*x$ denotes the elementary embedding from $\mathbb{U}$ into ${}^*\mathbb{U}$. We say "by transfer" to indicate the use of the elementary equivalence between $\mathbb{U}$ and ${}^*\mathbb{U}$. We say "by saturation" when using the saturation property of ${}^*\mathbb{U}$.

Let us enumerate some well-known facts of nonstandard topology. Let X be a topological space. The monad of $x \in X$ is $\mu(x) = \bigcap \{ {}^*U \mid x \in U \in \tau \}$, where τ is the topology of X. A subset U of X is open if and only if $\mu(x) \subseteq {}^*U$ for all $x \in U$. A subset F of X is closed if and only if $\mu(x) \cap {}^*F \neq \varnothing$ implies $x \in F$ for all $x \in X$. A subset K of X is compact if and only if for any $x \in {}^*K$ there is a $y \in K$ with $x \in \mu(y)$. A map $f : X \to Y$ of topological spaces is continuous at $x \in X$ if and only if for any $y \in \mu(x)$ we have ${}^*f(y) \in \mu(f(x))$.

Next, let X be a uniform space. Two points x, y of ${}^*X$ are said to be infinitely close, denoted by $x \approx y$, if for any entourage U of X we have $(x, y) \in {}^*U$. $\approx$ is an equivalence relation on ${}^*X$. The monad of $x \in X$ is equal to $\mu(x) = \{ y \in {}^*X \mid x \approx y \}$. Thus, in the case of uniform spaces, one can define the monad of $x \in {}^*X$. For each entourage U of X, the U-neighbourhood of $x \in X$ is $U[x] = \{ y \in X \mid (x, y) \in U \}$. A map $f : X \to Y$ of uniform spaces is uniformly continuous if and only if $x \approx y$ implies ${}^*f(x) \approx {}^*f(y)$ for all $x, y \in {}^*X$.

Let $\{ X_i \}_{i \in I}$ be a family of uniform spaces. Let P be the product $\prod_{i \in I} X_i$ of $\{ X_i \}_{i \in I}$, and let Q be the coproduct $\coprod_{i \in I} X_i$ of $\{ X_i \}_{i \in I}$. Let $\approx_X$ denote the "infinitely close" relation of a uniform space X. For any $x, y \in P$, $x \approx_P y$ if and only if $x(i) \approx_{X_i} y(i)$ for all $i \in I$. For any $x, y \in Q$, $x \approx_Q y$ if and only if there is an $i \in I$ such that $x, y \in X_i$ and $x \approx_{X_i} y$.

2. Definition of μ-homology theory

Let X be a uniform space and G an internal abelian group. We denote by $C_p X$ the internal free abelian group generated by ${}^*X^{p+1}$, and by $C_p(X; G)$ the internal abelian group of all internal homomorphisms from $C_p X$ to G. Each member of $C_p(X; G)$ can be represented in the form $\sum_{i=0}^n g_i \sigma_i$, where $\{ g_i \}_{i=0}^n$ is an internal hyperfinite sequence of members of G, and $\{ \sigma_i \}_{i=0}^n$ is an internal hyperfinite sequence of members of ${}^*X^{p+1}$. A member $(a_0, \dots, a_p)$ of ${}^*X^{p+1}$ is called a *p-microsimplex* if $a_i \approx a_j$ for all $i, j \le p$, or equivalently, $\mu(a_0) \cap \cdots \cap \mu(a_p) \neq \varnothing$. A member of $C_p(X; G)$ is called a *p-microchain* if it can be represented in the form $\sum_{i=0}^n g_i \sigma_i$, where $\{ g_i \}_{i=0}^n$ is an internal hyperfinite sequence of members of G, and $\{ \sigma_i \}_{i=0}^n$ is an internal hyperfinite sequence of p-microsimplices. We denote by $M_p(X; G)$ the subgroup of $C_p(X; G)$ consisting of all p-microchains. The boundary map $\partial_p : M_p(X; G) \to M_{p-1}(X; G)$ is defined by

$$\partial_p(a_0, \dots, a_p) = \sum_{i=0}^{p} (-1)^i (a_0, \dots, \hat{a}_i, \dots, a_p).$$

More precisely, we first define an internal map $\partial'_p : C_p(X; G) \to C_{p-1}(X; G)$ by the same equation. We see that $\partial'_p(M_p(X; G)) \subseteq M_{p-1}(X; G)$. ∂_p is defined as the restriction of ∂'_p to $M_p(X; G)$. Thus $M_\bullet(X; G)$ forms a chain complex.

Let $f : X \to Y$ be a uniformly continuous map. By the nonstandard characterization of uniform continuity, we see that for every p-microsimplex $(a_0, \dots, a_p)$ on X, $({}^*f(a_0), \dots {}^*f(a_p))$ is a p-microsimplex on Y. The induced homomorphism $M_\bullet(f; G) : M_\bullet(X; G) \to M_\bullet(Y; G)$ of f is defined by

$$M_p(f; G)(a_0, \dots, a_p) = ({}^*f(a_0), \dots {}^*f(a_p)).$$

Thus we have the functor $M_\bullet(\cdot; G)$ from the category of uniform spaces to the category of chain complexes. *μ-homology theory* is the composition of functors $H_\bullet(\cdot; G) = H_\bullet M_\bullet(\cdot; G)$, where $H_\bullet$ in the right hand side is the ordinary homology theory of chain complexes.