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Omega-limit sets for shift spaces and unimodal maps

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ABSTRACT

This paper investigates the structure of points $u \in \mathcal{A}^{\mathbb{N}}$ that are such that the omega-limit set $\omega(u, \sigma)$ is precisely X, where $X \subseteq \mathcal{A}^{\mathbb{N}}$ is an internally transitive shift space. We then use those results to study the possible structures of the omega-limit set of the turning point for a unimodal map. Examples are provided of unimodal maps f where no iterate of the turning point c is recurrent and $\omega(c, f)$ is either a minimal Cantor set or properly contains a minimal Cantor set.

1. Introduction

There is a great deal of literature focusing on the dynamics of a unimodal map of an interval to itself [5,7,9]. Of interest within this family of maps is the behavior of the map restricted to the omega-limit set of the turning point. This paper is motivated by the problem of characterizing when $\omega(c, f)$ is a Cantor set and the action $f|_{\omega(c,f)}$ is a minimal continuous map; for brevity we refer to this situation as one where $\omega(c, f)$ is a minimal Cantor set.

Because unimodal maps can be studied from a symbolic point of view (for example, in terms of itineraries of points and kneading sequences), in order to understand the omega-limit set of the turning point we may study omega-limit sets within the shift space $\{0, 1\}^{\mathbb{N}}$. In [1], the question was posed whether it is possible to have $\omega(c, f)$ a minimal Cantor set when every point in the orbit of c is non-recurrent. This question is equivalent to the problem of constructing a shift maximal sequence $u \in \{0, 1\}^{\mathbb{N}}$ such that $\omega(u, \sigma)$ is a

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minimal Cantor set but $\sigma^k(u) \notin \omega(u, \sigma)$ for all $k \in \mathbb{N}$. In investigating this question we became interested in further determining which shift spaces can be realized as the omega-limit set of the turning point for a unimodal map when the turning point is non-recurrent. Thus, the main theorems of this paper investigate the more general question of how to construct a given shift space $X \subseteq \mathcal{A}^{\mathbb{N}}$ as the omega-limit set of a point $u \in \mathcal{A}^{\mathbb{N}}$. In [2] it was shown that a shift space $X \subseteq \mathcal{A}^{\mathbb{N}}$ can be generated as the omega-limit set of a point $u \in \mathcal{A}^{\mathbb{N}}$ if and only if X is internally transitive; we provide a constructive characterization of the points $u \in \mathcal{A}^{\mathbb{N}}$ such that $\omega(u, \sigma) = X$.

In Section 2 we establish the terminology and notation that will be used throughout this paper. Section 3 investigates which shift spaces $X \subseteq \mathcal{A}^{\mathbb{N}}$ can be generated as the omega-limit set of a point $u \in \mathcal{A}^{\mathbb{N}}$ and characterizes the structure of u for a given shift space X. We then determine when u can be constructed as a point in X and when u can be constructed such that no shift of u is in X. In Section 4 we apply the results from Section 3 to gain a better understanding of the structure of the omega-limit set of the turning point for a unimodal map. Theorem 4.2 classifies those shift spaces X that are topologically conjugate to $\omega(c, f)$ for some unimodal map f, and in Theorem 4.5 it is shown that the set of parameters in the tent family for which $\omega(c, T)$ is a minimal Cantor set and $c_n \notin \omega(c, T)$ for all $n \in \mathbb{N}$ is dense in $[\sqrt{2}, 2]$. We conclude by providing examples of kneading sequences $\mathcal{K}(f)$ generated from the construction in Theorem 3.3 such that $\omega(c, f)$ is either a non-minimal Cantor set or the union of a Cantor set and a countable set when no iterate of the turning point is recurrent.

2. Background on shift spaces and sequences

Let $\mathcal{A}$ be a finite set of letters called an *alphabet*. A finite string of letters from $\mathcal{A}$ is called a *word*, and the set of all finite words over $\mathcal{A}$ is denoted $\mathcal{A}^\star$; for completeness, we allow $\emptyset$ to denote the empty word. We set $\mathcal{A}^{\mathbb{N}}$ to be the set of all one-sided infinite strings of letters from $\mathcal{A}$. Given a sequence $x = x_1x_2x_3 \cdots \in \mathcal{A}^{\mathbb{N}}$, the *shift map* $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is defined by $\sigma(x) = x_2x_3x_4 \cdots$. Let $\mathcal{A}$ be given the discrete metric topology and assign the product topology on $\mathcal{A}^{\mathbb{N}}$ by $d(x, y) = 1/2^{n-1}$ where n is the least number such that $x_1x_2 \cdots x_n \neq y_1y_2 \cdots y_n$; hence $\mathcal{A}^{\mathbb{N}}$ is a compact metrizable space. A subset $X \subseteq \mathcal{A}^{\mathbb{N}}$ is called a *shift space* if X is closed and X is strongly invariant, i.e. $\sigma(X) = X$.

Given a shift space X, we let the *language* $\mathscr{L} = \mathscr{L}(X)$ be the set of all words from $\mathcal{A}^\star$ that appear in X and the *forbidden words* $\mathscr{F} = \mathscr{F}(X)$ be the set of all words from $\mathcal{A}^\star$ that never appear in X. We denote by $\mathscr{F}' = \mathscr{F}'(X) \subseteq \mathscr{F}$ the set of *first offender words*, i.e. those words $F \in \mathscr{F}$ such that every proper subword of F is in $\mathscr{L}$. We additionally denote by $\mathscr{L}_n = \mathscr{L}_n(X)$ the set of all words from $\mathscr{L}$ of length $n \in \mathbb{N}$. We say that a shift space X is *transitive* if for every $u, v \in \mathscr{L}$ there exists a $w \in \mathscr{L}$ such that $uwv \in \mathscr{L}$.

We get examples of transitive minimal systems by considering substitution shift spaces. A *substitution* is a function $\theta : \mathcal{A} \to \mathcal{A}^\star \setminus \emptyset$ that is extended to $\mathcal{A}^\star$ or to $\mathcal{A}^{\mathbb{N}}$ by concatenation; that is, $\theta(xy) = \theta(x)\theta(y)$. A *fixed point* of a substitution is a sequence $u \in \mathcal{A}^{\mathbb{N}}$ such that $\theta(u) = u$. We take the closure of the shift orbit of the fixed point u to form a substitution subshift. For more information on substitutions see [8]. We will use substitutions to illustrate constructions in Section 4.

We now define some terminology that is standard among arbitrary continuous maps on compact metric spaces.

Given $f : E \to E$, a continuous map on a compact metric space, and a point $x \in E$, the *omega-limit set of x under f* is the set $\omega(x, f) = \{y \in E \mid \text{ there exists } n_1 < n_2 < \cdots \text{ with } f^{n_i}(x) \to y\}$. A point $x \in E$ is *recurrent* if for every open set U containing x, there exists $m \in \mathbb{N}$ such that $f^m(x) \in U$; equivalently, x is recurrent if and only if $x \in \omega(x, f)$. A point $x \in E$ is *uniformly recurrent* if for every open set U containing x, there exists an $M \in \mathbb{N}$ such that for all $j \geq 0$, $f^{j+k}(x) \in U$ for some $0 < k \leq M$. In terms of shift spaces, a sequence $w \in \mathcal{A}^{\mathbb{N}}$ is *recurrent* if every word u appearing in w appears infinitely often in w and is *uniformly recurrent* if for any word u appearing in w, there exists an M such that every word of length M in w contains at least one occurrence of u. When a word u has the property that it occurs in