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A theorem of Piccard's type and its applications to polynomial functions and convex functions of higher orders

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ABSTRACT

In the paper a theorem of Piccard's type is proved and, consequently, the continuity of $\mathcal{D}$-measurable polynomial functions of n-th order as well as $\mathcal{D}$-measurable n-convex functions is shown. The paper refers to the papers [6] and [9].

1. Introduction

In 2013 U.B. Darji defined a σ-ideal of "small" sets in an abelian Polish group which is equivalent to the notation of meager sets in a locally compact group. He was motivated by J.P.R. Christensen's paper [2] where the author defined *Haar null* sets in an abelian Polish group in such a way that in a locally compact group it is equivalent to the notation of Haar measure zero sets (see also [7,8,12]).

It is well known that there are some analogies between measure and category in locally compact groups (see [14]). It turns out (see [2,3,9]) that in non-locally compact abelian Polish groups Haar meager sets have an analogous behavior to Haar null sets; e.g.

- both families are invariant σ-ideals,

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- $0 \in \operatorname{int}(A-A)$ for each universally measurable non-Haar null set as well as each Borel non-Haar meager set A (it constitutes the *Steinhaus dichotomy* formulated by N.H. Bingham and A.J. Ostaszewski in [1]),
- each compact set is Haar null and Haar meager,
- each set containing a translation of each compact set is non-Haar meager and non-Haar null.

Moreover, the definition of Christensen measurability introduced by P. Fischer and Z. Słodkowski in [5] seems to be analogous to the definition of $\mathcal{D}$-measurability from [9]; e.g.

- each Christensen measurable or $\mathcal{D}$-measurable homomorphism is continuous (which constitutes the *Darboux dichotomy* from [1]),
- each Christensen measurable or $\mathcal{D}$-measurable t-Wright convex function is continuous

(see [5,9,10,13]).

Here we prove a theorem of Piccard's type[1] and next we show that each $\mathcal{D}$-measurable function which is n-convex with respect to the cone S is continuous, as well as each $\mathcal{D}$-measurable polynomial function of n-th order is continuous. Analogous results for Christensen measurable function have been obtained by Z. Gajda in [6].

2. Preliminaries on $\mathcal{D}$-measurability

The following fact will be useful in the sequel:

Theorem 1. [4, p. 90] *If X is an abelian Polish group, there exists an equivalent complete metric ρ on X, which is invariant; i.e. $\rho(x,y) = \rho(x+z, y+z)$ for every $x, y, z \in X$.*

Let us recall some necessary definitions and theorems from [3] and [9].

Definition 1. [3, Definition 2.1] Let X be an abelian Polish group. A set $A \subset X$ is *Haar meager* iff there is a Borel set $B \subset X$ with $A \subset B$, a compact metric space K and a continuous function $f : K \to X$ such that $f^{-1}(B+x)$ is meager in K for all $x \in X$. The family of all Haar meager sets in X we denote by $\mathcal{HM}$.

By the definition we can easily see that the family of all Haar meager sets is invariant, i.e. $A + x \in \mathcal{HM}$ for each $x \in X$ and $A \subset X$.

Theorem 2. [3, Theorem 2.2, Theorem 2.9] *In each abelian Polish group the family $\mathcal{HM}$ is a σ-ideal contained in the σ-ideal of all meager sets.*

Remark 1. In [3] Darji proved that in a locally compact abelian Polish group the family of all Haar meager sets and the family of all meager sets are equivalent, but in each non-locally compact abelian Polish group it is not true.

Definition 2. [9, Definition 1] Let X be an abelian Polish group. A set $A \subset X$ is *$\mathcal{D}$-measurable* iff $A = B \cup M$ for a Haar meager set $M \subset X$ and a Borel set $B \subset X$.

Theorem 3. [9, Theorem 6] *The family $\mathcal{D}$ of all $\mathcal{D}$-measurable sets in each abelian Polish group is a σ-algebra.*

[1] We call it in this way because a consequence of this theorem is the fact that $0 \in \operatorname{int}(A-A)$; cf. [11, Theorem 2.9.1].