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On the deleted squares of lens spaces ☆

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ABSTRACT

The configuration space $F_2(M)$ of ordered pairs of distinct points in a manifold M, also known as the deleted square of M, is not a homotopy invariant of M: Longoni and Salvatore produced examples of homotopy equivalent lens spaces M and N of dimension three for which $F_2(M)$ and $F_2(N)$ are not homotopy equivalent. They also asked whether two arbitrary 3-dimensional lens spaces M and N must be homeomorphic in order for $F_2(M)$ and $F_2(N)$ to be homotopy equivalent. We give a partial answer to this question using a novel approach with the Cheeger–Simons differential characters.

1. Introduction

The configuration space $F_n(M)$ of ordered n-tuples of pairwise distinct points in a manifold M is a much studied classic object in topology. Until a few years ago, it was conjectured that homotopy equivalent manifolds M must have homotopy equivalent configuration spaces $F_n(M)$. Much had been done towards proving this conjecture until in 2004 Longoni and Salvatore [8] found a counterexample using the non-homeomorphic but homotopy equivalent lens spaces $L(7,1)$ and $L(7,2)$. They proved that the configuration spaces $F_2(L(7,1))$ and $F_2(L(7,2))$ are not homotopy equivalent by showing that their universal covers have different Massey products: all of the Massey products vanish for the former but not for the latter.

This result prompted a natural question pertaining specifically to lens spaces: does the homotopy type of configuration spaces distinguish all lens spaces up to homeomorphism? This question was studied by Miller [9] for two–point configuration spaces, also known under the name of deleted squares. Miller extended

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the Massey product calculation of [8] to arbitrary lens spaces; however, comparing the resulting Massey products turned out to be too difficult and the results proved to be inconclusive.

In this paper, we study the above question using Cheeger–Simons flat differential characters [3]. With their help, we obtain new algebraic restrictions on possible homotopy equivalences between deleted squares of lens spaces, see Theorem 2 below. In particular, these restrictions allow for a much easier comparison of the Massey products, producing multiple examples of the pairs of homotopy equivalent lens spaces whose deleted squares are not homotopy equivalent, see Section 7.3. Unlike the examples of Longoni and Salvatore [8], many of our examples have non-vanishing sets of Massey products on both of their universal covers. In fact, with computer assistance, we have been able to conclude the following.

Theorem 1. *For any prime $p \le 23$ the configuration spaces $F_2(L(p,q))$ and $F_2(L(p,q'))$ are homotopy equivalent if and only if the lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic.*

It remains to be seen if our techniques are sufficient to answer the question in general.

Here is an outline of the paper. The bulk of it deals with Cheeger–Simons flat differential characters. Given a lens space $L(p,q)$, denote by X_0 its configuration space $F_2(L(p,q))$. We study the Cheeger–Simons character **cs** which assigns to each representation $\alpha : \pi_1(X_0) \to SU(2)$ a homomorphism $\mathbf{cs}(\alpha) : H_3(X_0) \to \mathbb{R}/\mathbb{Z}$. This homomorphism is obtained by realizing each of the generators of $H_3(X_0)$ by a continuous map $f : M \to X_0$ of a closed oriented 3-manifold M and letting $\mathbf{cs}(\alpha)$ of that generator equal the Chern–Simons function of the pull-back representation $f^*\alpha$.

The realization problem at hand is known to have a solution due to an abstract isomorphism $\Omega_3(X_0) = H_3(X_0)$, where $\Omega_3(X_0)$ is the third oriented cobordism group of X_0; however, explicit realizations $f : M \to X_0$ have to be constructed by hand. Using the naturality of **cs**, we reduce this task to a somewhat easier problem of realizing homology classes in $H_3(L(p,q) \times L(p,q))$ and solve it by finding a set of generators realized by Seifert fibered manifolds. The Chern–Simons theory on such manifolds is sufficiently well developed for us to be able to finish the calculation of **cs**. This calculation leads to the following theorem, which constitutes the main technical result of the paper.

Theorem 2. *Let p be an odd prime and assume that the deleted squares X_0 and X_0' of lens spaces $L(p,q)$ and $L(p,q')$ are homotopy equivalent. Then there exists a homotopy equivalence $f : X_0' \to X_0$ such that, with respect to the canonical generators of the fundamental groups, the homomorphism $f_* : \pi_1(X_0') \to \pi_1(X_0)$ is given by a scalar matrix $\operatorname{diag}(\alpha,\alpha)$, where $\pm q' = q\alpha^2 \pmod p$.*

The homotopy equivalence $f : X_0' \to X_0$ of this theorem lifts to a homotopy equivalence $\tilde{f} : \tilde{X}_0' \to \tilde{X}_0$ of the universal covering spaces of the type studied by Longoni and Salvatore [8] and Miller [9]. The homotopy equivalence $\tilde{f}$ naturally possesses equivariance properties made explicit by the knowledge of the induced map f_* on the fundamental groups. In the rest of the paper, we use these properties to reduce the comparison problem for the Massey products on $\tilde{X}_0'$ and $\tilde{X}_0$ to an algebraic problem in certain cyclotomic rings arising as cohomology of $\tilde{X}_0$ and $\tilde{X}_0'$. This is still a difficult problem, which is solved in individual examples with the help of a computer.

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