Topology and its Applications 209 (2016) 153–169

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Rational spaces in resolving classes

Timothy L. Clark

Department of Mathematics, Western Michigan University, Kalamazoo, MI 49006, United States

ARTICLE INFO

Article history:
Received 8 January 2016
Received in revised form 4 June 2016
Accepted 9 June 2016
Available online 14 June 2016

MSC:
11E04
55P62
55U30

Keywords:
Homotopy theory
Rational homotopy theory
Closed classes
Resolving classes
The Witt ring

ABSTRACT

A class of pointed spaces is called a resolving class if it is closed under weak equivalences and pointed homotopy limits. Let $\mathcal{R}(A)$ denote the smallest resolving class containing a space A. We say X is A-resolvable if X is in $\mathcal{R}(A)$, which induces a partial order on the category pointed spaces. We develop an algebraic criterion for determining if X is A-resolvable when X and A are rational spaces. The goal of this work is to develop some understanding of the structure of this resolvability relation, and in particular to use our algebraic criterion to help us better understand the rational case.

1. Introduction

The complexity of the Dror Farjoun cellular lattice [1] has proven to be an interesting area of investigation, with various efforts being made to illuminate its structure. Hess and Parent, for example, have shown that the sublattice of rational spaces admits an embedding of a certain quotient of the Witt group [2]. The definition of this lattice relies on the idea of closed classes, the dual of which are referred to as resolving classes [3]. Unsurprisingly, resolving classes lead us to a relation on spaces in a similar manner to the cellular lattice induced by closed classes.

Resolving classes appear as a focal point in Strom's homotopy-theoretical proof of Miller's theorem [4], and have found applications in Schwass's work on phantom maps [5]. However, they remain considerably less studied and less utilized than their dual counterparts. This paper is a first step in hunting down some dual results from the theory pertaining to the cellular lattice, as well as an attempt at expanding the current repertoire of results concerning resolving classes in general. The hope is that, as the theory surrounding resolving classes develops, they begin to find more applications in other areas of research.

E-mail address: timothy.l.clark@wmich.edu.

http://dx.doi.org/10.1016/j.topol.2016.06.007
0166-8641/

The paper is organized as follows:

Section 2 is dedicated to preliminaries, and includes overviews of Moore–Postnikov towers, rational homotopy theory, and rational quadratic forms.

Section 3 is used to establish some general results about resolving classes. The main result here is:

Theorem 12. *Let $m, n \in \mathbb{N}$ and suppose $\mathcal{R}$ is a resolving class. The following are equivalent:*

- $K(V, n) \in \mathcal{R}$ *for some nonzero* $\mathbb{F}$*-vector space* V*;*
- $K(W, m) \in \mathcal{R}$ *for all* $\mathbb{F}$*-vector spaces* W *and all* $m \leq n$*.*

Theorem 12 is used in understanding A-resolvable spaces where A is a rational space. Being able to form $K(V, n)$ for arbitrary vector spaces V allows us to form certain rational spaces X from their Postnikov tower using $K(\pi_k X, n)$.

Section 4 is dedicated to expanding our understanding of the rational case, the main results being:

Theorem 13. *For (simply connected) rational spaces X and A:*

- *if the anticonnectivity of X is strictly less than the anticonnectivity of A then $X \in \mathcal{R}(A)$;*
- *if $X \in \mathcal{R}(A)$, then the anticonnectivity of X is less than or equal to the anticonnectivity of A;*
- *$X \in \overline{\mathcal{R}}(A)$ if and only if the anticonnectivity of X is less than or equal to the anticonnectivity of A.*

Theorem 15. *Let X and A be rational spaces of anticonnectivity $n+1$. Then $X \in \mathcal{R}(A)$ if and only if there is a homotopy class*

$$f : X \to \prod_{i \in \mathcal{I}} A$$

for some index set $\mathcal{I}$, with $\pi_n(f)$ injective.

Theorem 16. *Let X and A be rational spaces with anticonnectivity $n+1$ and minimal Sullivan models $(\Lambda V, d)$ and $(\Lambda W, d)$ respectively. Then $X \in \mathcal{R}(A)$ if and only if there is a CDGA morphism*

$$\phi : \coprod_{i \in \mathcal{I}} (\Lambda W, d) \to (\Lambda V, d)$$

for some index set $\mathcal{I}$, with the linear part of ϕ surjective in degree n.

Theorem 13 implies, for rational spaces X and A, the question of whether or not X is A-resolvable can be answered easily if the spaces have unequal anticonnectivities. The answer in the case of equal anticonnectivity is Theorem 15, which is dual to a result known as the Chachólski–Parent–Stanley criterion (Theorem 1 from [2]). We offer a proof of Theorem 15, but to the best of the author's knowledge, a proof of the Chachólski–Parent–Stanley criterion does not yet exist in the literature. Theorem 16 is the algebraic translation of Theorem 15 which amplifies the theorem's utility as the algebraic criterion can sometimes be much easier to check. The section ends with some examples of using the algebraic criterion to determine some conditions for a rational space to be resolvably equivalent to an Eilenberg–MacLane space or (rationalized) sphere.

Section 5 concerns the resolvability relation in general and deals with proving that the same quotient of the Witt group that Hess and Parent found within the cellular lattice [2] can be uncovered within the resolvability relation as well. This is achieved by constructing CDGA's whose differentials correspond to