Topology and its Applications 209 (2016) 170–180

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Ideal structure of the classical ring of quotients of $C(X)$

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ARTICLE INFO

Article history:
Received 8 April 2016
Received in revised form 9 June 2016
Accepted 11 June 2016
Available online 16 June 2016

MSC:
54C40

Keywords:
Classical ring of quotient
Maximal z°-ideal
Basically disconnected space
Quasi-F-space
Q-point

ABSTRACT

In this paper we show that the set of z-ideals and the set of z° ideals (=d-ideals) of the classical ring of quotients $q(R)$ ($q(X)$) of a reduced ring R with property A ($C(X)$) coincide. Using this fact, we observe that each maximal ideal of $q(R)$ is the extension of a maximal z°-ideal of R. The members of maximal z°-ideals of $C(X)$ contained in a given maximal ideal are topologically characterized and using this, it turns out that the extension $\mathcal{O}^p$ of each O^p, $p \in \beta X$ is a maximal ideal of $q(X)$ if and only if X is a basically disconnected space. Topological spaces X are also characterized for which every $\mathcal{O}^p$ is contained in a unique maximal ideal of $q(X)$ and in this case, the maximal ideals of $q(X)$ are characterized. Finally, using the concept of z-ideal in $q(X)$, we characterize the regularity of $q(X)$. For instance, we observe that $q(X)$ is regular if and only if for each $f \in C(X)$, there exists a regular (non-zero divisor) element r such that $Z(f) \cap \operatorname{coz} r$ is open in $\operatorname{coz} r$ or equivalently, $\frac{|f|}{|r|}$ is an idempotent in $q(X)$.

1. Introduction

In this paper we denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real valued continuous functions on a completely regular Hausdorff space X and all rings in this article are assumed to be commutative rings with identity. We recall that a ring satisfies property A if every finitely generated ideal consisting of zero divisors has a nonzero annihilator and it said to be reduced if it has zero nilradical. Clearly, $C(X)$ is a reduced ring with property A. An element r of a ring R is called *regular* (nonzero divisor) if $ra = 0$, $a \in R$ implies that $a = 0$. An ideal of a ring is called *regular* if it contains a regular element, otherwise, whenever an ideal consists entirely of zero divisors, then it is called a *non-regular* ideal. We denote by $r(X)$ the set of all regular elements of $C(X)$ and it is easy to see that $f \in r(X)$ if and only if $\operatorname{int}_X Z(f) = \emptyset$ or equivalently, $\operatorname{coz} f$ is dense in X, where the *zero-set* $Z(f)$ is the set of zeros of f and $X \setminus Z(f) = \operatorname{coz} f$ is the *cozero-set* of f. The classical ring of quotients $q(R)$ of a ring R is the ring of all equivalence classes of

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http://dx.doi.org/10.1016/j.topol.2016.06.008

formal fractions $\frac{a}{r}$, for $a, r \in R$, where r is regular and equivalence relation is the obvious one. We will use the following well known result in the next section.

Lemma 1.1 *([11], Corollary 2.6). A ring R has property A if and only if $q(R)$ has.*

For each a in a ring R, the intersection of all maximal (minimal prime) ideals of R containing a is denoted by M_a (P_a). An ideal I in a ring R is said to be a z-ideal (z°-ideal) if for each $a \in I$, we have $M_a \subseteq I$ ($P_a \subseteq I$). The ideals M_a and P_a, for each $a \in R$, are called *basic z-ideals* and *basic z°-ideals* respectively. For each $f \in C(X)$, it is easy to see that $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ and $P_f = \{g \in C(X) : \mathrm{int}_X Z(f) \subseteq \mathrm{int}_X Z(g)\}$, see also [3] and [4]. Using these equalities, an ideal I in $C(X)$ is a z-ideal (z°-ideal) if and only if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$ ($\mathrm{int}_X Z(f) \subseteq \mathrm{int}_X Z(g)$) imply that $g \in I$. Now it is clear that every z°-ideal in $C(X)$ is a z-ideal and for an arbitrary ring we have the following result.

Lemma 1.2 *([2], Corollary 1.10). The Jacobson radical of a ring R is zero if and only if every z°-ideal is a z-ideal.*

For each $p \in \beta X$, where βX is the *Stone–Čech compactification* of X, the ideal M^p (resp., O^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{cl}_{\beta X} Z(f)$ (resp., $p \in \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z(f)$). By Theorem 7.3 in [8], each maximal ideal of $C(X)$ is precisely of the form M^p, $p \in \beta X$ and each O^p is contained in the unique maximal ideal M^p, see Theorem 7.13 in [8]. In case $p \in X$, the ideals M^p and O^p are denoted by M_p and O_p respectively. Whenever a z°-ideal is maximal in the realm of z°-ideals, we call it a *maximal z°-ideal*. By the following lemma, every non-regular maximal ideal in a reduced ring with property A is a maximal z°-ideal.

Lemma 1.3 *([2], Corollary 1.22). If R is a reduced ring with property A, then every maximal ideal consisting only of zero divisors is a z°-ideal.*

If R and S are rings and $\varphi : R \to S$ is a ring homomorphism, then for each ideal J in S, the ideal $\varphi^{-1}(J)$ is called the *contraction* of J to R and is denoted by J^c. Whenever I is an ideal of R, then the ideal I^e generated by $\varphi(I)$ is called the *extension* of I to S. When we talk about "contraction" and "extension" in this paper, we mean the natural homomorphism φ from R ($C(X)$) to its classical ring of quotients $q(R)$ ($q(X)$) with $\varphi(a) = \frac{a}{1}$ ($\varphi(f) = \frac{f}{1}$), where $a \in R$ ($f \in C(X)$). By the following lemma, each z°-ideal of $q(R)$ contracts to z°-ideal of R.

Lemma 1.4 *([2], Corollary 1.8). If R is a reduced ring and $f : R \to S^{-1}R$ is the natural ring homomorphism, then every z°-ideal of $S^{-1}R$ contracts to a z°-ideal of R.*

A point $p \in \beta X$ is called an *F-point* if the ideal O^p is a prime ideal in $C(X)$ and a space X is called an *F-space* if every point of βX is an F-point. It is well known that X is an F-space if and only if every finitely generated ideal of $C(X)$ is principal, see Theorem 14.25 in [8]. A space is said to be *basically disconnected* if the closure of every cozero-set of the space is open in the space. For undefined terms and notations, we refer the reader to [6–8] and [11].

2. z-ideals of classical rings of quotients

In this section, we show that for a reduced ring R with property A, the set of z-ideals and the set of z°-ideals of the classical ring of quotients $q(R)$ of R coincide. Moreover, if R is a reduced ring, we also observe in this section that there is a one-to-one correspondence between z°-ideals of R and z°-ideals of $q(R)$. First we characterize the maximal ideals of $q(R)$ in terms of maximal z°-ideals of R, for a reduced ring R with property A.