Topology and its Applications 209 (2016) 207–219

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Parity biquandle invariants of virtual knots

Aaron Kaestner [a], Sam Nelson [b,*,1], Leo Selker [b]

[a] Department of Mathematics, 3225 West Foster Avenue, Chicago, IL 60625-4895, United States
[b] Department of Mathematical Sciences, Claremont McKenna College, 850 Columbia Ave., Claremont, CA 91711, United States

ARTICLE INFO

Article history:
Received 13 October 2015
Received in revised form 15 June 2016
Accepted 15 June 2016
Available online 23 June 2016

MSC:
57M27
57M25

Keywords:
Virtual knots
Parity biquandles
Cocycle invariants
Enhancements of counting invariants

ABSTRACT

We define counting and cocycle enhancement invariants of virtual knots using parity biquandles. The cocycle invariants are determined by pairs consisting of a biquandle 2-cocycle ϕ^0 and a map ϕ^1 with certain compatibility conditions leading to one-variable or two-variable polynomial invariants of virtual knots. We provide examples to show that the parity cocycle invariants can distinguish virtual knots which are not distinguished by the corresponding non-parity invariants.

1. Introduction

In 1996, Louis Kauffman introduced the world to *virtual knot theory* in [11]. Virtual knot theory is a combinatorial generalization of the theory of knotted curves in $\mathbb{R}^3$, now known as *classical knot theory*. Each ambient isotopy class of knotted oriented curves in $\mathbb{R}^3$ coincides with an equivalence class of combinatorial objects known as *signed Gauss diagrams*; however, the set of all such equivalence classes includes classes which do not correspond to classical knots. These extra classes are known as *virtual knots*. Virtual knots can be understood geometrically as knots in certain 3-manifolds ($\Sigma \times [0, 1]$ for Σ an orientable surface) up to equivalence by stabilization of Σ [4].

* Corresponding author.
E-mail addresses: amkaestner@northpark.edu (A. Kaestner), sam.nelson@cmc.edu (S. Nelson), lselker13@gmail.com (L. Selker).

1 Partially supported by Simons Foundation collaboration grant 316709.

http://dx.doi.org/10.1016/j.topol.2016.06.010
0166-8641/

In [11], it is observed that every classical knot is represented by a diagram in which every crossing is *evenly intersticed*, i.e. every crossing has an even number of over- and under-crossing points along the knot between its over and under instances. In virtual knots, a crossing can have an even or odd number of crossing points between its over and under instances, and moreover this even or odd *parity* is not changed by Reidemeister moves. In [13], parity was used to create a number of invariants for virtual knots. In [7] the notion of parity was generalized to integer-valued maps and used to define new invariants of virtual knots. Parity invariants are very good at distinguishing classical knots from non-classical virtual knots as well as simply distinguishing virtual knot types.

In [8] (and see also [12]), algebraic structures known as *biquandles* were introduced. Given any finite biquandle X, there is a non-negative integer-valued invariant of oriented knots and links known as the *biquandle counting invariant* which counts homomorphisms from the fundamental biquandle of a knot K to X, represented as colorings of the semiarcs of K by elements of X. Cocycles in the second cohomology of a finite biquandle were first used to enhance the biquandle counting invariant in [3].

In [10], biquandles incorporating the notion of parity were introduced (see also [1]). In this paper we extend the counting invariant to the case of finite parity biquandles and define enhancements of the counting invariant using *parity enhanced cocycles*, cocycles in the second cohomology of the even part of the parity biquandle with extra information analogous to the virtual cocycles in [6].

The paper is organized as follows. In Section 2 we review the basics of biquandles. In Section 3 we review parity biquandles and introduce the parity biquandle counting invariant. In Section 4 we review biquandle cohomology and define parity cocycle enhancements of the counting invariant. We provide examples demonstrating that the parity enhanced cocycle invariants are stronger than the corresponding unenhanced cocycle invariants and the corresponding non-parity invariants for virtual knots. In Section 5 we conclude with questions for future research.

2. Biquandles

A biquandle is an algebraic structure with axioms motivated by the Reidemeister moves (see [8,12] etc.). It can be defined abstractly:

Definition 1. A *biquandle* is a set X along with two operators, $\overline{\triangleright}$ and $\underline{\triangleright}$, both maps $X \times X \mapsto X \times X$, such that:

(i) For all $x \in X$, $x \,\overline{\triangleright}\, x = x \,\underline{\triangleright}\, x$.
(ii) We have right invertibility of both maps and pairwise invertibility, i.e. the maps $\alpha_y : x \mapsto x \,\overline{\triangleright}\, y$, $\beta_y : x \mapsto x \,\underline{\triangleright}\, y$, and $S : (x, y) \mapsto (y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$ are all invertible.
(iii) For all $x, y, z \in X$, we have the *exchange laws*:

$$(z \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (x \,\overline{\triangleright}\, y) = (z \,\overline{\triangleright}\, x) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, x)$$

$$(x \,\overline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\overline{\triangleright}\, y) = (x \,\underline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z)$$

$$(y \,\underline{\triangleright}\, x) \,\underline{\triangleright}\, (z \,\overline{\triangleright}\, x) = (y \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (x \,\underline{\triangleright}\, z)$$

Example 1. A well-known type of biquandle is the *Alexander biquandle*. The biquandle's underlying set X is a module over the ring $\Lambda = \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ of two-variable Laurent polynomials. In particular, note that s and t are invertible, so for Alexander biquandles structures on finite rings or fields (where s and t are elements of the ring), the characteristic must be relatively prime to s and t. The operations are defined as: