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On a Lichnerowicz type cohomology attached to a function

Cristian Ida

Department of Mathematics and Computer Science, University Transilvania of Braşov, Braşov 500091, Str. Iuliu Maniu 50, Romania

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ABSTRACT

In this paper we define a new cohomology of a smooth manifold called Lichnerowicz type cohomology attached to a function. Firstly, we study some basic properties of this cohomology as: a de Rham type isomorphism, dependence on the function, singular forms, relative cohomology, Mayer–Vietoris sequence, homotopy invariance and next, a regular case is considered. The notions are introduced using techniques from the study of two cohomologies of a smooth manifold: the Lichnerowicz cohomology and the cohomology attached to a function.

1. Introduction

Let us consider a n-dimensional smooth manifold M and θ a closed 1-form on M. Denote by $\Omega^r(M)$ the set of all r-differential forms on M and consider the twisted operator $d_\theta : \Omega^r(M) \to \Omega^{r+1}(M)$ defined by $d_\theta = d - \theta\wedge$, where d is the usual exterior derivative. Since $d\theta = 0$, we easily obtain $d_\theta^2 = 0$. The differential complex $(\Omega^\bullet(M), d_\theta)$ is called the *Lichnerowicz complex* of M; its cohomology groups $H_\theta^\bullet(M)$ are called the *Lichnerowicz cohomology groups* of M. This is the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [14] or Lichnerowicz–Jacobi cohomology on Jacobi and locally conformal symplectic geometry manifolds, see [1,12,13], and Novikov in the context of Hamiltonian mechanics [17]. It is also known in literature as Morse–Novikov cohomology and plays an important role when studying the geometry, topology and Morse theory of the underlying manifold M, see for instance [19–22]. We also notice that Vaisman [26], studied it under the name of "adapted cohomology" in the context of locally conformally Kähler manifolds. Locally, the Lichnerowicz cohomology complex becomes the de Rham complex after a change $\varphi \mapsto e^{-f}\varphi$ with f a smooth function which satisfies $df = \theta$, that is d_θ is the unique differential in $\Omega^\bullet(M)$ which makes the multiplication by the smooth function e^{-f} an isomorphism of cochain complexes

E-mail address: cristian.ida@unitbv.ro.

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$e^{-f}:(\Omega^\bullet(M),d_\theta)\to(\Omega^\bullet(M),d)$. In the case when θ is exact, there is a related differential $\delta_{t,f}=d+tdf\wedge$ ($t\in\mathbb{R}$ and f is a Morse function on M) introduced in [30] by Witten in order to obtain an analytic proof of the Morse inequalities, see also [31]. The cohomology of the deformed Witten differential is very useful in the study of the topology of the fiber $f^{-1}(c)$, see [6,7].

Other important applications of the Lichnerowicz cohomology appear when studying locally (globally) conformal structures: (co)symplectic, (co)Kähler, Jacobi, Dirac etc., and in many cases it is an invariant at conformal changes. For more about this cohomology see for instance [1,9,13,25,26,29]. Also, such a cohomology can be generalized in the context of Jacobi algebroid, that is a pair (A,θ) where $A=(A,[\cdot,\cdot]_A,\rho_A)$ is a Lie algebroid over a manifold M and $\theta\in\Gamma(A^*)$ is a 1-cocycle, i.e. $d_A\theta=0$, see [3,8,11]. Then the twisted cohomological operator is given by $d_{A,\theta}=d_A-\theta\wedge$, and the Lichnerowicz cohomology of (M,θ) is just the Jacobi Lie algebroid cohomology of usual Lie algebroid $(TM,[\cdot,\cdot],Id)$ with the 1-cocycle θ.

On the other hand, in [16], Monnier gave the definition and basic properties of a new cohomology of a manifold, called *cohomology attached to a function*. The definition is the following: If f is a smooth function on a smooth manifold M, then we can define the linear operator $d_f:\Omega^r(M)\to\Omega^{r+1}(M)$ by

$$d_f\varphi=fd\varphi-rdf\wedge\varphi\,,\ \forall\,\varphi\in\Omega^r(M).$$

It is easy to see that $d_f^2=0$, and so, we have a differential complex $(\Omega^\bullet(M),d_f)$ which is called the *differential complex attached to the function* f of M; its cohomology groups $H^\bullet_f(M)$ are called the *cohomology groups attached to the function* f of M. This cohomology was considered for the first time in [15] in the context of Poisson geometry, and more generally, Nambu–Poisson geometry.

The main difference between the operators d_f and d_θ is given by the fact that d_f is an antiderivation, i.e.

$$d_f(\varphi\wedge\psi)=d_f\varphi\wedge\psi+(-1)^{\deg\varphi}\varphi\wedge d_f\psi,\ \forall\,\varphi,\psi\in\Omega^\bullet(M),\tag{1.1}$$

while d_θ is not an antiderivation, and it satisfies

$$d_\theta(\varphi\wedge\psi)=d\varphi\wedge\psi+(-1)^{\deg\varphi}\varphi\wedge d_\theta\psi,\ \forall\,\varphi,\psi\in\Omega^\bullet(M).\tag{1.2}$$

Moreover, according with [16], the d_f-cohomology is just the Lie algebroid cohomology associated with a certain Lie algebroid structure $([\cdot,\cdot]_f,\rho_f)$ on TM, where the anchor is defined by $\rho_f(X)=fX$ and the Lie bracket is defined by $[X,Y]_f=(1/f)[fX,fY]$. Thus, d_f-cohomology is a Lie algebroid cohomology, while d_θ-cohomology is a Jacobi algebroid cohomology.

In this paper we introduce a generalization of the usual Lichnerowicz cohomology starting from the cohomology attached to a function. We observe that if θ is a closed one form on an arbitrary smooth manifold M and $f\in C^\infty(M)$ then $d_f(f\theta)=0$, that is $(TM,f\theta)$ is a Jacobi algebroid where $(TM,[\cdot,\cdot]_f,\rho_f)$ is the above Lie algebroid. Then, it is natural to consider the cohomological differential operator $d_{f,\theta}$: $\Omega^r(M)\to\Omega^{r+1}(M)$ defined by

$$d_{f,\theta}\varphi=d_f\varphi-f\theta\wedge\varphi\,,\ \ \varphi\in\Omega^r(M),\tag{1.3}$$

which satisfies $d_{f,\theta}\circ d_{f,\theta}=0$. Thus, we obtain the differential complex $(\Omega^\bullet(M),d_{f,\theta})$ which is called the *Lichnerowicz type complex attached to the function* f of M and its cohomology groups $H^\bullet_{f,\theta}(M)$ are called the *Lichnerowicz type cohomology groups attached to the function* f of M. This is just the Jacobi algebroid cohomology of $(TM,f\theta)$ with the Lie algebroid structure $([\cdot,\cdot]_f,\rho_f)$ and, using some arguments as in the study of d_f-cohomology, the main goal of this paper is to study the main properties of this new cohomology of a smooth manifold.

The paper is organized as follows: In the second section we make some remarks about our cohomology when the one form θ is exact and we relate it with globally conformal cosymplectic manifolds and with