Topology and its Applications 209 (2016) 239–262

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Cyclically presented groups, lower central series and line arrangements

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ARTICLE INFO

Article history:
Received 22 August 2014
Received in revised form 16 June 2016
Accepted 17 June 2016
Available online 21 June 2016

Keywords:
Group theory
Lower central series
Line arrangements
Graded Lie algebra
Fundamental group

ABSTRACT

The quotients G_k/G_{k+1} of the lower central series of a finitely presented group G are an important invariant of this group. In this work we investigate the ranks of these quotients in the case of a certain class of cyclically presented groups, which are groups generated by $x_1, \dots, x_n$ and having only cyclic relations:

$$x_{i_t} x_{i_{t-1}} \cdot \ldots \cdot x_{i_1} = x_{i_{t-1}} \cdot \ldots \cdot x_{i_1} x_{i_t} = \cdots = x_{i_1} x_{i_t} \cdot \ldots \cdot x_{i_2}.$$

Using tools from group theory and from the theory of line arrangements we explicitly find these ranks, which depend only at the number and length of these cyclic relations. It follows that for these groups the associated graded Lie algebra $gr(G)$ decomposes, in any degree, as a direct product of local components.

1. Introduction

Let G be a group. Its lower central series is defined by $G_1 = G, G_k = [G_{k-1}, G], k \geq 2$. The graded sum

$$gr(G) \doteq \bigoplus_{k\geq 1} G_k/G_{k+1}$$

has a graded Lie algebra structure induced by the commutator bracket on G. In general, $gr(G)$ reflects many properties of the group G. For example, if G is finitely generated, then the abelian groups G_k/G_{k+1} are also finitely generated. Although there is a straightforward algorithm to find out the structure of G_k/G_{k+1}, determining the dimensions symbolically for a parameterized family of groups might be exceedingly difficult. Thus, determining this structure, or even the ranks of the quotients $\phi_k(G) \doteq \mathrm{rank}(G_k/G_{k+1})$, for a certain class of finitely generated groups, is a significant problem.

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http://dx.doi.org/10.1016/j.topol.2016.06.013

In the case of $G = \mathbb{F}_n$, the free group of rank n, Magnus (see e.g. [16]) showed that $gr(\mathbb{F}_n)$ is the free Lie algebra on n generators, whose ranks were computed by Witt [25]. Hall [12] introduced the basic commutators of $\mathbb{F}_n$, showing that the coset classes of weight k form a basis of G_k/G_{k+1}. Therefore, for the free group, the structure of $gr(\mathbb{F}_n)$ is completely known. Another well-known example is the work of Chen [1], computing the structure of $gr(\mathbb{F}_n/(\mathbb{F}_n)'')$.

In this paper we concentrate on a particular class of finitely presented groups, originating from fundamental groups of complements of hyperplane arrangements. These groups have a presentation where all the relations are cyclic relations; that is, of the form:

$$x_{i_t}^{s_t} x_{i_{t-1}}^{s_{t-1}} \cdot \ldots \cdot x_{i_1}^{s_1} = x_{i_{t-1}}^{s_{t-1}} \cdot \ldots \cdot x_{i_1}^{s_1} x_{i_t}^{s_t} = \cdots = x_{i_1}^{s_1} x_{i_t}^{s_t} \cdot \ldots \cdot x_{i_2}^{s_2},$$

where $\{x_1, \ldots, x_n\}$ are the generators of the group, $2 \leq t$, $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, \ldots, n\}$ is an increasing subsequence of indices and $s_j \in \langle x_1, \ldots, x_n \rangle$. We call such groups *hyperplane* groups (see Definition 2.1) and the above relation a *cyclic relation* of length t. When all the conjugating elements are equal to the identity, i.e. $s_j = e$, then we call such a group *cyclically-presented* (see Definition 2.8). The simplest example of such a group is the group $\mathbb{F}_n/\langle R_n \rangle$ generated by n generators with one cyclic relation of length n:

$$R_n: \ x_n x_{n-1} \cdot \ldots \cdot x_1 = x_{n-1} x_{n-2} \cdot \ldots \cdot x_1 x_n = \cdots = x_1 x_n \cdot \ldots \cdot x_2.$$

Obviously, $\mathbb{F}_n/\langle R_n \rangle \simeq \mathbb{F}_{n-1} \oplus \mathbb{Z}$, and thus the structure of $gr(\mathbb{F}_n/\langle R_n \rangle)$ is known.

Another example of a hyperplane group is the pure braid group PB_n (see [17, Theorem 2.3] for a proper presentation of this group). The study of the lower central series quotients of the pure braid group was initiated by Kohno [14]. Given an arrangement of hyperplanes $\mathcal{A} = \{H_1, \ldots, H_n\} \subset \mathbb{C}^m$, the fundamental group $\pi_1(\mathbb{C}^m - \mathcal{A})$ is a hyperplane group (see Remark 2.2); thus, from this perspective, it is clear why the pure braid group is a hyperplane group, as it is the fundamental group of the complement of a hyperplane arrangement known as the braid arrangement. In general, if W is a real reflection group, $\mathcal{A}_W \subset \mathbb{C}^m$ the associated hyperplane arrangement and B_W the Artin group associated to W, then $\pi_1(\mathbb{C}^m - \mathcal{A}_W) \simeq \ker(B_W \twoheadrightarrow W)$ is a hyperplane group.

As an arrangement of lines in $\mathbb{C}^2$ is an example of a hyperplane arrangement, for any line arrangement $\mathcal{L}$ the associated fundamental group $G = \pi_1(\mathbb{C}^2 - \mathcal{L})$ is a hyperplane group. Kohno [13] proved, using the mixed Hodge structure on $H^1(\mathbb{C}^2 - \mathcal{L}, \mathbb{Q})$, that $gr(G) \otimes \mathbb{Q}$ is isomorphic to the nilpotent completion of the holonomy Lie algebra of $\mathbb{C}^2 - \mathcal{L}$. Falk proved [6], using Sullivan's 1-minimal models, that the lower central series ranks $\phi_k(G)$ are determined only by the combinatorics of $\mathcal{L}$. However, a precise combinatorial formula for $\phi_k(G)$, and even for $\phi_3(G)$, is not known (for the general formula for ϕ_3, see e.g. [22, Corollary 3.6] or Remark 3.11 below).

These motivations lead us to investigate these ranks for hyperplane and cyclically-presented groups. While it is fairly easy to find out the rank ϕ_2 for any hyperplane group (see Section 3.1.1), it is a harder task when considering the rank ϕ_3. The main result of this paper is that for a certain class of cyclically-presented groups, there is a tight, combinatorial upper bound on ϕ_3 (see Proposition 3.10 and Theorem 3.15). Explicitly, for such a group G, using only group theoretic arguments, we prove that $\phi_3(G) \leq \sum_{i \geq 3} n_i \phi_3(\mathbb{F}_{i-1})$, where n_i is the number of cyclic relations of length i of G.

We then associate to this group G a line arrangement $\mathcal{L}(G)$, such that $\pi_1(\mathbb{C}^2 - \mathcal{L}(G)) \simeq G$. By [7], given any line arrangement $\mathcal{L}$, $\phi_3(\pi_1(\mathbb{C}^2 - \mathcal{L}))$ is bounded from below by the above upper bound. Thus, for this class of cyclically-presented groups, one can calculate ϕ_3 directly, i.e. the third quotient of the lower central series behaves as if the group is a direct product of free groups. From this it follows, by [21], that these cyclically-presented groups are *decomposable* (see Definition 4.17), that is, we prove another main result: the quotient G_k/G_{k+1} decomposes, for $k \geq 2$, as a direct product of local components and that $\phi_k(G) = \sum_{i \geq 3} n_i \phi_k(\mathbb{F}_{i-1})$ for every $k \geq 2$ (see Theorem 4.16). Hence, for this class of groups, we have found the complete structure of the additive groups of $gr(G)$.