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Box resolvability

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ABSTRACT

We say that a topological group G is partially box κ-resolvable if there exist a dense subset B of G and a subset A of G, $|A| = \kappa$ such that the subsets $\{aB : a \in A\}$ are pairwise disjoint. If $G = AB$ then G is called box κ-resolvable. We prove two theorems. If a topological group G contains an injective convergent sequence then G is box ω-resolvable. Every infinite totally bounded topological group G is partially box n-resolvable for each natural number n, and G is box κ-resolvable for each infinite cardinal $\kappa, \kappa < |G|$.

1. Introduction

For a cardinal κ, a topological space X is called *κ-resolvable* if X can be partitioned into κ dense subsets [1]. In the case $\kappa = 2$, these spaces were defined by Hewitt [6] as *resolvable spaces*. If X is not κ-resolvable then X is called *κ-irresolvable*.

In topological groups, the intensive study of resolvability was initiated by the following remarkable theorem of Comfort and van Mill [3]: every countable non-discrete Abelian topological group G with finite subgroup $B(G)$ of elements of order 2 is 2-resolvable. In fact [19], every infinite Abelian group G with finite $B(G)$ can be partitioned into ω subsets dense in every non-discrete group topology on G. On the other hand, under MA, the countable Boolean group G, $G = B(G)$ admits maximal (hence, 2-irresolvable) group topology [8]. Every non-discrete ω-irresolvable topological group G contains an open countable Boolean subgroup provided that G is Abelian [11] or countable [18], but the existence of non-discrete ω-irresolvable group topology on the countable Boolean group implies that there is a P-point in ω^* [11]. Thus, in some models of ZFC (see [14]), every non-discrete Abelian or countable topological group is ω-resolvable. We

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mention also κ-resolvability of every infinite totally bounded topological group G of cardinality κ [9]. For systematic exposition of resolvability in topological and left topological group see [4, Chapter 13].

This note is to introduce a more delicate kind of resolvability, the box resolvability.

Given a group G and a cardinal κ, we say that a subset B of G is a *partial box* of index κ, if there exists a subset A of G, $|A| = \kappa$ such that the subsets $\{aB : a \in A\}$ are pairwise disjoint. In addition, if $G = AB$ then B is called a *box* of index κ. Example: a subgroup H of G is a box of index $|G : H|$, and any set R of representatives of right cosets of G by H is a box of index $|H|$.

We also use the factorization terminology [16]. For subset A, B of G, the product AB is called a *partial factorization* if $aB \bigcap a'B = \emptyset$ for any distinct $a, a' \in A$. If $G = AB$ then the product AB is called a *factorization* of G. Thus, a box B of index κ is a right factor of some factorization $G = AB$ such that $|A| = \kappa$.

We say that a topological group G is *(partially) box* κ-resolvable if there exists a (partial) box B of index κ dense in G. Clearly, every partially box κ-resolvable topological group is κ-resolvable, but a κ-resolvable group needs not to be box κ-resolvable (see Examples 1 and 2). However, I do not know, whether every 2-resolvable group is partially box 2-resolvable.

On exposition: in section 2, we prove two theorems announced in Abstract and discuss some prospects of box resolvability in section 3.

2. Results

We begin with two examples demonstrating purely algebraic obstacles to finite box resolvability.

Example 1. We assume that the group $\mathbb{Z}$ of integer numbers is factorized $\mathbb{Z} = A + B$ so that A is finite, $|A| > 1$. By the Hajós theorem [5], B is periodic: $B = m + B$ for some $m \neq 0$. Then $m\mathbb{Z} + B = B$ and $m\mathbb{Z} + b \subseteq B$ for $b \in B$.

Now we endow $\mathbb{Z}$ with the topology τ of finite indices (having $\{n\mathbb{Z} : n \in \mathbb{N}\}$ as the base at 0). Since $m\mathbb{Z}$ is open in τ and $|A| > 1$, we conclude that B is not dense, so $(\mathbb{Z}, \tau)$ is box n-irresolvable for each $n > 1$. □

Example 2. Every torsion group G without elements of order 2 has no boxes of index 2. We assume the contrary: $G = B \bigcup gB$, $B \bigcap gB = \emptyset$ and $e \in B$, e is the identity of G. Then $g^2 B = B$ and B contains the subgroup $< g^2 >$ generated by g. Since g is an element of odd order, we have $g \in < g^2 >$ and $g \in B \bigcap gB$. □

Let G be a countable group. Applying [13, Theorem 2], we can find a factorization $G = AB$ such that $|A| = |B| = \omega$. Hence, if we endow G with a group topology τ, there are no algebraic obstacles to box ω-resolvability of (G, τ).

In what follows, we use two elementary observations. Let G be a topological group, H be a subgroup G and R be a system of representatives of right cosets of G by H. Let AB be a factorization of H. Then we have

(1) If B is dense in H then $A(BR)$ is a factorization of G with dense BR;
(2) If R is dense in G then $A(BR)$ is a factorization of G with dense BR.

Example 3. Let G be a non-discrete metrizable group and let A be a subgroup of G. If A is either finite or countable discrete then there is a factorization AB of G such that B is dense in G.

In view of (1), we may suppose that G is countable. Let $\{U_n : n \in \omega\}$ be a base of topology of G. For each $n \in \omega$, we choose $x_n \in U_n$ so that $Ax_n \bigcap Ax_m = \emptyset$ if $n \neq m$. Then we complement the set $\{x_n : n \in \omega\}$ to some full system B of representatives of right cosets of G by A. □