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On the non-existence of torus actions

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ABSTRACT

In this paper we compute the higher left derived functors of the indecomposable functor, in certain degrees, for a general class of algebras. The techniques do not depend on the existence of a Projective extension sequence—a common method used to make such computations throughout the literature and as a result, we generalize all known computations of these higher derived functors. As a result of these calculations we have the following applications. First, we prove that certain torus actions on a Quasitoric manifold are restricted due to the combinatorial structure of the orbit implying the non-existence of certain torus actions. Second, we obtain a refinement of the class of equivariantly formal torus actions, splitting them into two classes. In the context of augmented simplicial algebras over a field, the results provide explicit computations of the cotangent complex. We also show very explicitly how one higher derived functor depends on another and in the case of Toric Topology, the generators of these derived functors are linked backed to the combinatorics of the orbit.

1. Introduction

Computing the homotopy groups of a Quasitoric manifold and related toric spaces – Borel space and Moment angle complex – is a challenging problem. General results, in particular, high dimensional calculations are sparse within the literature. In earlier work, Buchstaber and his collaborators used cellular-type arguments along with the notion of neighborliness of the orbit space to make a specific high dimensional homotopy calculation [11]. These results gave a particular homotopy group of the Moment angle complex in terms of the combinatorics of the orbit, and this could, by way of a certain fibration, give information on the higher homotopy groups of a Quasitoric manifold. There were also π_2 calculations and other miscellaneous computations for the homotopy of the Borel space (which for a Quasitoric manifold M endowed with a torus action is $ET \times_T M$). In this paper, we are primarily concerned with a Quasitoric manifold M. Specific

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calculations require that we write $M(\lambda)$ and we will do so when necessary or convenient. We briefly recall, λ is a map of integer lattices that depends on the orbit space [11,16], satisfying certain conditions – what is commonly referred to as condition $(*)$. There are some interesting observations that arise by way of the long exact sequence of homotopy groups induced from the fibration $M \to ET \times_T M \to BT$ where BT is the classifying space. For instance, by leveraging methods from Toric Topology, there are specific homotopy calculations for $\mathbb{C}P^n \sharp \mathbb{C}P^n$ and other complicated spaces, at least from a π_*-computational perspective. Various homotopy-theoretic calculations are strewn throughout the literature, but many of them are collected in [1] and [11].

Before moving on to unstable spectral sequences, we bring to the reader's attention research in the stable range, specifically, splittings and other decompositions. There is an ongoing and active program being undertaken by Fred Cohen, Martin Bendersky and their collaborators in regards to stable splittings coming from the polyhedral functor. This line of inquiry is described in [8], among others by the said authors. In this setting, many toric spaces can be decomposed into a certain suspension of a wedge which can under certain limited cases, shed insight into homotopy calculations.

Another approach to obtaining homotopy information for these Toric spaces and in particular, Quasitoric manifolds, is to set-up and compute, if possible, certain unstable spectral sequences. When referring to these spectral sequences, we are primarily concerned with Toric-Topological applications [1].

For a complex orientable theory E, there is a certain Composite Functor Spectral Sequence (CFSS) [9]. The E_∞-term of this spectral sequence is the associated graded group of the E_2-term of the Unstable Adams Novikov Spectral Sequence [9]. In regards to the CFSS, one part of the input are the higher right derived functors of the primitive element functor $R^iPE_*(M)$ where $E_*(M)$ is the E-homology of a Quasitoric manifold M thought of as a certain coalgebra. There are the usual complications associated to such objects and the determination of the spectral sequence; namely, computing the E_2-term, computing the differentials and resolving any potential extension issues. However, there are additional difficulties, the least of which is computing $R^iPE_*(M)$, which is a very challenging problem. It is also related, as we will elaborate below – in the Main Results section – to a variety of other deep problems, such as computing the cotangent complex, in a certain setting. In the context of Toric Topology there are more complications, such as the complexity of the cohomology ring of quasitoric manifolds—a free algebra modulo a complicated ideal. As a consequence, many of the tools break-down at various stages because most of these spectral sequences work "reasonably" well for spheres and other related spaces such as loops on spheres, but the cohomology ring of these spaces are less complicated than that of a Quasitoric manifold. More specifically, the algebra structure is simpler. Assuming these challenges can be met, there is the task of computing the unstable coaction on the E-homology for these manifolds in the requisite categories—unstable $\mathcal{U}$-comodules and unstable $\mathcal{G}$-coalgebras [9]. We will not stress these comodule structures here, but the interested reader can refer to [1] for coaction formulae.

There are methods to compute these higher derived functors, chief among them is the long exact sequence of $R^iPE_*(-)$ induced from an *injective extension sequence*, which is, roughly speaking, a short exact sequence in the category of coalgebras over a graded ring [12]. Since coalgebras are often less intuitive, there is the dual notion of *projective extension sequence* where one speaks of the induced long exact sequence of the higher left derived functors of the indecomposable functor Q i.e., $L_iQE^*(-)$. If the algebras are of finite type, then one computation can be determined from another. Once again, there is the problem that the conditions required to be projective extension sequence fail almost always, except in the most limited circumstances. That is, the projective or more generally, the freeness condition does not hold. The current theory does not address this issue in a manner that makes explicit computations feasible in our setting. However, the results in this paper do not require any such conditions to hold and we are able to make very explicit computations in a very general setting.

We briefly summarize our approach. The chain complex used to compute the higher left derived functors $L_iQE^*(-)$ is written out explicitly and we develop a bookkeeping mechanism to keep track of cycles as