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Characterizing a class of completable fuzzy metric spaces

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In this paper we give a characterization of the class of completable strong (non-Archimedean) fuzzy metric spaces, in the sense of George and Veeramani.

1. Introduction

The problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several authors from different points of view. Here we use the concept of fuzzy metric space that George and Veeramani [1,3] introduced and studied with the help of continuous t-norms. In [2,6], it is proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable spaces. This result allows to restate some classical theorems on metrics in the realm of fuzzy metric spaces. Nevertheless, the theory of fuzzy metric completion is, in this context, very different from the classical theory of metric

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completion. Indeed, Gregori and Romaguera proved that there exist fuzzy metric spaces which are not completable [7]. Later, the same authors gave a characterization of those fuzzy metric spaces that are completable, which we reformulate, for our convenience, as follows:

Theorem 1.1. *([8]) A fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.
(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.
(c3) *The assignment* $t \to \lim_n M(a_n, b_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$, *provided with the usual topology of* $\mathbb{R}$.

In the literature, there were examples of non-completable strong fuzzy metrics that do not satisfy (c1) or (c2) [7,8], and recently [4], the authors have constructed a non-completable fuzzy metric space which does not satisfy (c3).

In this paper we first observe that (c1)–(c3) constitute an independent axiomatic system and then we will proof, after several lemmas, that strong fuzzy metrics satisfy (c3), or in other words (Theorem 4.7): A strong fuzzy metric space $(X, M, *)$ is completable if and only if M satisfies (c1) and (c2). Several corollaries can be obtained from this theorem, for instance a characterization of completable fuzzy ultrametrics (Corollary 4.9) and also we could obtain that metric spaces admit a unique completion, but we do not insist on it because it is well known from the properties of the standard fuzzy metric. Several examples illustrate our results.

The structure of the paper is as follows. After the preliminaries section, in Section 3 we prove that (c1)–(c3) constitute an independent axiomatic system. In Section 4 we give a characterization for the class of completable strong fuzzy metrics.

2. Preliminaries

Definition 2.1. (George and Veeramani [1]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$
(GV2) $M(x, y, t) = 1$ if and only if $x = y$
(GV3) $M(x, y, t) = M(y, x, t)$
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

Remark 2.2. $M(x, y, \_)$ is non-decreasing for all $x, y \in X$.

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$