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Extreme points and geometric aspects of compact convex sets in asymmetric normed spaces ☆

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ABSTRACT

The Krein–Milman theorem states that every compact convex subset in a locally compact convex space is the closure of the convex hull of its extreme points. Inspired by this result, we investigate the existence of extreme points in compact convex subsets of asymmetric normed spaces. We focus our attention in the finite dimensional case, giving a geometric description of all compact convex subsets of a finite dimensional asymmetric normed space.

1. Introduction

An *asymmetric normed space* is a real vector space X equipped with a so called *asymmetric norm* q. This means that $q : X \to [0, \infty)$ is a function satisfying

1. $q(tx) = tq(x)$ for every $t \geq 0$ and $x \in X$,
2. $q(x + y) \leq q(x) + q(y)$ and
3. $q(x) = 0 = q(-x)$ if and only if $x = -x = 0$.

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Any asymmetric norm induces a non-symmetric topology on X that is generated by the asymmetric open balls $B_q(x, \varepsilon) = \{y \in X \mid q(y - x) < \varepsilon\}$. This topology is a T_0 topology in X for which the vector sum on X is continuous. However, in general this topology is not even Hausdorff and the scalar multiplication is not continuous. Thus (X, q) fails to be a topological vector space.

Any asymmetric normed space (X, q) has an associated (symmetric) norm. That is the norm $q^s : X \to [0, \infty)$ defined by the formula:

$$q^s(x) = \max\{q(x), q(-x)\}.$$

Compactness on these spaces has been widely studied and nowadays there are some very interesting results that describe the general structure of compact sets in asymmetric normed spaces (see [1,4,5]). When the linear space has finite (linear) dimension, it is known that the separation axiom T_1 implies T_2 —and thus normed— and therefore all the properties concerning the topology of those spaces are perfectly known. The general case is then when the topology has no other separation axiom than T_0.

An asymmetric normed lattice is a classic example of a non-Hausdorff asymmetric normed space. Namely, if $(X, \|\cdot\|, \leq)$ is a normed lattice, there is a canonical way to define an asymmetric norm in X by means of the formula

$$q(x) := \|x \vee 0\|, \quad x \in X.$$

In this case the asymmetric norm q is called an *asymmetric lattice norm* and the pair (X, q) is an *asymmetric normed lattice*. This special kind of asymmetric normed spaces is important and interesting, mainly by the applications in theoretical computer science, and particularly in complexity theory (see, e.g. [6]).

An interesting problem related with compactness in asymmetric normed spaces is the existence of a so called center. Namely, we say that K' is a *center* for a compact subset K of an asymmetric normed space (X, q) if K' is q^s-compact (compact in the topology generated by the norm q^s) and

$$K' \subset K \subset K' + \theta(0)$$

where $\theta(0) = \{x \in X \mid q(x) = 0\}$. A q-compact set with a center is called *strongly q-compact* (or simply, *strongly compact*). It is well known that not all q-compact sets in an asymmetric normed space are strongly compact (see [1, Example 12] and [4, Example 4.6]). However, in certain cases the existence of a center characterizes the q-compactness [1, Section 5]. Moreover, in [4] it was proved that strong compactness and compactness coincide in the class of q^s-closed compact sets in a finite dimensional asymmetric normed lattice. Furthermore, it was proved in [7] that every q-compact convex set in a 2-dimensional asymmetric normed lattice is strongly compact (even if it is not q^s-closed). This result shows that convexity plays an important role while working with compact sets in asymmetric normed spaces.

In relation to convexity, a big effort has been made to translate the classic results of functional analysis to the non-asymmetric case (see [3]). One of these results is the Krein–Milman Theorem. It states that every compact convex subset of a locally convex space is the closure of the convex hull of its extreme points. In particular, each compact convex subset of a locally convex space has at least an extreme point. Concerning the asymmetric case, it was proved in [3] that every Hausdorff asymmetric normed space satisfies the Krein–Milman Theorem.

However, in the general case this result is not longer true, not even in finite dimensional asymmetric normed spaces. For example, let us consider the asymmetric norm $|\cdot|_a : \mathbb{R} \to [0, \infty)$ in $\mathbb{R}$ given by $|t|_a = \max\{0, t\}$. The set $(-1, 1]$ is a compact convex set and its only extreme point is 1. Thus $(-1, 1]$ cannot be the convex hull of its extreme points. Even more, the closure of $\{1\}$ (in the asymmetric topology of $\mathbb{R}_a$) coincides with the interval $[1, \infty)$, which is far from being $(-1, 1]$.