Topology and its Applications 203 (2016) 22–31

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A new approach on Lipschitz compact operators

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ARTICLE INFO

Article history:
Received 17 September 2014
Received in revised form 27 November 2014
Accepted 29 November 2014
Available online 2 February 2016

MSC:
46E10
46E15
46J10

Keywords:
Lipschitz operator
Lipschitz p-summing operator
Lipschitz p-integral operator
Lipschitz-free Banach space

ABSTRACT

The notions of Lipschitz-free compact and Lipschitz-free weakly compact operators between metric spaces are introduced. Some nonlinear versions of Schauder's theorem and Gantmacher's theorem on compact and weakly compact linear operators are proved. The Davis–Figiel–Johnson–Pełczyński factorization theorem is stated for Lipschitz-free weakly compact operators.

0. Introduction

In this note we deal with Lipschitz-free compact and Lipschitz-free weakly compact operators between metric spaces. They are nonlinear versions of the notions of compact and weakly compact linear operators between Banach spaces. We give several characterizations of Lipschitz-free compact and Lipschitz-free weakly compact operators. These results are nonlinear versions of the classical theorems due to Schauder and Gantmacher on compact and weakly compact linear operators, respectively. We also obtain a version for Lipschitz-free weakly compact operators of the factorization theorem of W.J. Davis et al. [6]. Similar versions of these results were stated for Banach-valued Lipschitz operators in [10]. The relationships between different classes of Lipschitz operators are studied. The key tool to obtain our results is a process of linearization of Lipschitz mappings provided by the Lipschitz-free space over a pointed metric space. We dedicate the following section to recall this process and present some known classes of Lipschitz operators.

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http://dx.doi.org/10.1016/j.topol.2015.12.072

1. Preliminaries

A pointed metric space X is a metric space with a base point that we always will represent by 0. If X is a normed space, 0 will be its origin. We denote by d the distance in any metric space.

Let X and Y be pointed metric spaces. Let us recall that a map $f: X \to Y$ is Lipschitz if there exists a real constant $C \geq 0$ such that $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. The infimum of such constants is denoted by $\mathrm{Lip}(f)$. In other words,

$$\mathrm{Lip}(f) = \sup\left\{\frac{d(f(x), f(y))}{d(x, y)} : x, y \in X,\ x \neq y\right\}.$$

We denote by $\mathrm{Lip}_0(X, Y)$ the set of all Lipschitz maps f from X into Y such that $f(0) = 0$. The elements of $\mathrm{Lip}_0(X, Y)$ are also referred to as Lipschitz operators. If E is a Banach space over the field $\mathbb{K}$ of real or complex numbers, $\mathrm{Lip}_0(X, E)$ is a Banach space with the Lipschitz norm Lip. The space $\mathrm{Lip}_0(X, \mathbb{K})$ is known as the Lipschitz dual of X and denoted frequently by $X^\#$.

The Lipschitz-free Banach space $\mathcal{F}(X)$ over a pointed metric space X is the closed linear span in $(X^\#)^*$ of the evaluation functionals $\delta_x: X^\# \to \mathbb{K}$ with $x \in X$, where

$$\delta_x(f) = f(x) \qquad \left(f \in X^\#\right).$$

This space was called and denoted so by G. Godefroy and N.J. Kalton in [9]. We refer to Weaver's book [16] for a complete study about spaces of Lipschitz functions.

Notation. Let E and F be Banach spaces. We denote by $\mathcal{L}(E, F)$ the Banach space of all bounded linear operators from E into F with the usual norm. $\mathcal{K}(E, F)$ and $\mathcal{W}(E, F)$ stand for the spaces of compact and weakly compact linear operators from E into F, respectively. As is customary, E^* stands for the dual space of E, B_E for the closed unit ball of E and J_E for the canonical isometric embedding from E into E^{**}. Given $M \subset E$, we denote by $\overline{\Gamma}(M)$ the closed, convex, balanced hull of M in E. For any $T \in \mathcal{L}(E, F)$, T^* denotes the adjoint operator of T from F^* into E^*.

We gather in the next theorem some properties of the Lipschitz-free space over a pointed metric space.

Theorem 1.1. *Let X and Y be pointed metric spaces.*

(i). *The Dirac map $\delta_X: X \to \mathcal{F}(X)$ given by $\delta_X(x) = \delta_x$ is a (nonlinear) isometry.*
(ii). *$\mathcal{F}(X)^*$ is isometrically isomorphic to $X^\#$ via the evaluation map $Q_X: X^\# \to \mathcal{F}(X)^*$ given by $Q_X(g)(\gamma) = \gamma(g)$ for all $g \in X^\#$ and $\gamma \in \mathcal{F}(X)$.*
(iii). *The closed unit ball of $\mathcal{F}(X)$ is the closed, convex, balanced hull in $(X^\#)^*$ of the set*

$$\left\{\frac{\delta_x - \delta_y}{d(x, y)} : x, y \in X,\ x \neq y\right\}.$$

(iv). *For each $f \in \mathrm{Lip}_0(X, Y)$, the Lipschitz adjoint map $f^\#: Y^\# \to X^\#$, given by $f^\#(g) = gf$ for all $g \in Y^\#$, is a continuous linear operator and $\left\|f^\#\right\| = \mathrm{Lip}(f)$.*
(v). *For each $f \in \mathrm{Lip}_0(X, Y)$, there exists a unique operator $L_f \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $(L_f)^* = Q_X f^\#(Q_Y)^{-1}$. Furthermore, $\|L_f\| = \mathrm{Lip}(f)$.*
(vi). *For each $f \in \mathrm{Lip}_0(X, Y)$, there exists a unique operator $L_f \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $L_f \delta_X = \delta_Y f$, that is, the following diagram commutes:*