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A new contribution to the fixed point theory in partial quasi-metric spaces and its applications to asymptotic complexity analysis of algorithms

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ABSTRACT

In this paper we continue the study of fixed point theory in partial quasi-metric spaces and its usefulness in complexity analysis of algorithms. Concretely we prove two new fixed point results for monotone and continuous self-mappings in 0-complete partial quasi-metric spaces and, in addition, we show that the assumptions in the statement of such results cannot be weakened. Furthermore, as an application, we present a quantitative fixed point technique which is helpful for asymptotic complexity analysis of algorithms.

1. Introduction

In 1970, D.S. Scott introduced fixed point techniques in order to describe the meaning of recursive specifications in denotational semantics for programming languages [18]. Since then a few attempts to reconcile the Scott model with metric spaces have been made (see, for instance, [19]). In this direction, in 1994, S.G. Matthews introduced the notion of partial metric space and developed a partial metric fixed point

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technique helpful to analyze the program correctness of those algorithms that use a recursive denotational specification in the spirit of Scott [9]. Later on, in 1995, M.P. Schellekens applied the ideas from Scott's mathematical theory of computation to asymptotic complexity analysis of algorithms. Specifically, he introduced a quasi-metric fixed point technique useful for discussing the asymptotic complexity classes of those algorithms whose running time of computing satisfies a recurrence equation [17]. It must be stressed that the Scott fixed point technique is qualitative while the Matthews–Schellekens techniques are quantitative. In 2006, H.P.A. Künzi, H. Pajooshesh and Schellekens introduced the notion of partial quasi-metric space in order to present a generalization of both partial metric spaces and quasi-metric spaces under the same framework [8]. Although the aforementioned metric fixed point techniques (by Matthews and Schellekens) seemed to have no connection beyond their origins, the partial quasi-metric spaces have allowed to show that the partial metric fixed point technique of Matthews can be applied to complexity analysis in the Schellekens sense recently (see [4] and [1]).

Inspired by the latest references, in this paper we focus our study on fixed point theory in partial quasi-metric spaces and its utility in complexity analysis of algorithms. Concretely, we develop a new quantitative fixed point technique in 0-complete partial quasi-metric spaces, which preserves the spirit of the partial metric ones for denotational semantics, and apply it to determine the asymptotic complexity class of algorithms whose running time of computing satisfies a recurrence equation. To this end, the remainder of the paper is organized as follows: Section 2 is devoted to recall those notions that will be crucial in order to present our new results. In Section 3 we present the new fixed point results and provide examples showing that the assumptions in the statement of such results cannot be weakened. Moreover, the relationship between our outcomes and a recent one from [10] is given in the same section. Finally, Section 4 is devoted to present the aforesaid quantitative fixed point technique, based on the results in Section 3, and its utility in asymptotic complexity analysis of algorithms.

2. Mathematical preliminaries

In the following we recall a few concepts which will be useful along the paper.

As usual [6], a partial order on a nonempty set X is a reflexive, antisymmetric and transitive binary relation $\leq$ on X. A partially ordered set is a pair $(X, \leq)$ such that X is nonempty set and $\leq$ is a partial order on X. Moreover, given a subset $Y \subseteq X$, an upper bound for Y in $(X, \leq)$ is an element $x \in X$ such that $y \leq x$ for all $y \in Y$. The supremum for Y in $(X, \leq)$, if exists, is an element $z \in X$ which is an upper bound for Y and, in addition, satisfies that $z \leq x$ provided that $x \in X$ is an upper bound for Y. We will denote by $\uparrow x$ ($\downarrow x$), with $x \in X$, the set $\{y \in X : x \leq y\}$ ($\{y \in X : y \leq x\}$). Furthermore, an element $x \in X$ is maximal (minimal) in $(X, \leq)$ provided that if there exists $y \in X$ such that $x \leq y$ ($y \leq x$), then $y = x$. A sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \leq)$ is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integer numbers.

Following [16], an element x of a partially ordered set $(X, \leq)$ is said to be a post-fixed (pre-fixed) point, with respect to $\leq$, of a mapping f from $(X, \leq)$ into itself provided that $x \leq f(x)$ ($f(x) \leq x$). As usually, a mapping f from a partially ordered set $(X, \leq)$ into itself will be called monotone if $f(x) \leq f(y)$ whenever $x \leq y$.

From now on, let us denote by $\mathbb{R}^+$ the set of nonnegative real numbers.

On account on [7, Chapter 11], a quasi-metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X.