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Chaos for linear fractional transformations of shifts ☆

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ABSTRACT

We characterize chaos for $\varphi(B)$ on Banach sequence spaces, where φ is a Linear Fractional Transformation and B is the usual backward shift operator. Characterizations are computable since they involve only the four complex numbers defining φ.

Chaos has been usually considered a non-linear phenomena, although linear chaos may appear if we let the underlying space to be infinite dimensional. According to Devaney [1] and [23], a continuous map on a metric space is chaotic provided it is topologically transitive and it has a dense set of periodic points. This paper deals with the chaotic behaviour of a class of bounded linear maps (operators) defined on separable Banach spaces. Within this context, it is well known that topological transitivity is equivalent to the existence of a dense orbit, and this property is known for operators as hypercyclicity. Therefore, we have that operators defined on separable Banach spaces are chaotic if and only if they are hypercyclic and admit a dense set of periodic points. The monographs [2,3] are very good sources for the theory of linear dynamics.

The first example of a hypercyclic operator on a Banach space was given by Rolewicz in 1969 [4]. He proved that multiples λB of the backward shift operator $B(x_1, x_2, \dots) := (x_2, x_3, \dots)$ are hypercyclic on the space ℓ^1 of absolutely summable sequences if and only if $|\lambda| > 1$; in fact, they are chaotic. Since then, (weighted) shift operators defined on sequence spaces have become a usual ground to study linear dynamics. Hypercyclicity for weighted backward shifts defined on ℓ^p was characterized by Salas and he also showed

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that any perturbation of the identity by a weighted backward shift is always hypercyclic [5]. Following this direction, characterizations for hypercyclic and chaotic weighted backward shifts defined on general sequence spaces were obtained in [6,7], and characterizations for chaotic perturbations of the identity by weighted backward shifts are also in [7]. Other results for the linear dynamics of operators of the form $P(T)$, where T is an operator and $P(z)$ is a polynomial or a more general function, can be found in [8–13].

DeLaubenfels and Emamirad [10] proved that for a given non-constant polynomial $P(z)$, $P(B)$ is chaotic on ℓ^p, $1 \leq p < \infty$, whenever $P(\mathbb{D})$ intersects the unit circle [10, Th. 2.8], where $\mathbb{D}$ is the open unit disc of $\mathbb{C}$. Besides, for $a, b \in \mathbb{K}$, $\mathbb{K}$ being the scalar field, they showed that $aI + bB$ is chaotic if and only if $|b| > |1 - |a||$. Further sufficient conditions for chaos of $P(B)$ in terms of the coefficients of the polynomial had been stated in [9]. The aim of this note is to keep going on the search of 'computable' conditions for chaos of $\varphi(B)$, where $\varphi(z)$ is an analytic function. In this case, we deal with Linear Fractional Transformations, that is, analytic functions of the form $\varphi(z) = (az + b)/(cz + d)$. We prove that $\varphi(B)$ is chaotic on ℓ^p if and only if

$$\left| |d|^2 - |c|^2 - \left| b\bar{d} - a\bar{c} \right| \right| < |bc - ad| ,$$

which obviously generalizes the above mentioned result in [10]. As a consequence, we also characterize chaotic weighted backward shifts on weighted ℓ^p spaces, result that serves us to know which operators of the form $\varphi(D)$ are chaotic on certain Banach spaces of analytic functions, where D is the differentiation operator. Although it will be shown later, we would like to point out that all operator considered in this paper are in fact (upper triangular) Toeplitz operators.

Let us start setting our frame of work. For a strictly positive weight sequence $v := (v_n)_n$, let

$$\ell^p(v) := \left\{ (x_n)_n \in \mathbb{C}^{\mathbb{N}},\ \|x\|^p := \sum_{n=1}^{\infty} |x_n|^p \, v_n < \infty \right\},\ 1 \leq p < \infty,$$

be the associated weighted ℓ^p-space. It is easy to check that condition

$$\sup_i \frac{v_i}{v_{i+1}} < \infty \tag{1}$$

characterizes boundedness of $B : \ell^p(v) \to \ell^p(v)$. This condition will always be assumed to hold. If the weighting sequence v coincides with the sequence of ones, the corresponding space will be denoted as ℓ^p.

Our aim is to state the chaotic behaviour of $\varphi(B)$ on ℓ^p, where φ is a Linear Fractional Transformation. In order to avoid trivial cases when $\varphi(z)$ reduces to a constant or to a degree 1 polynomial, we will assume that $ad - cb \neq 0$ and $c \neq 0$. Therefore, by saying that φ is a Linear Fractional Transformation (LFT for short) we mean

$$\varphi(z) = \frac{az + b}{cz + d},\ a, b, c, d \in \mathbb{C},\ ad - cb \neq 0,\ c \neq 0. \tag{2}$$

There are several ways to describe how the operator $\varphi(B)$ is defined but we will only speak here about two of them. The first one is to recall that the spectrum of B (i.e., the set of $\lambda \in \mathbb{C}$ such that $\lambda I + B$ is not invertible) is the closed unit disc $\overline{\mathbb{D}}$. Now, if we impose that $|d/c| > 1$ we have $\varphi(B) = (aB + bI)(cB + dI)^{-1}$ is a well defined bounded operator on ℓ^p. Our second approach defining $\varphi(B)$ is to use the Taylor expansion (around the origin) of φ, which turns out to be

$$\frac{b}{d} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{ad - bc}{cd} \left(\frac{c}{d} \right)^n z^n.$$