Topology and its Applications 203 (2016) 98–107

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On paracompactness, completeness and boundedness in uniform spaces ☆

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A R T I C L E   I N F O

Article history:
Received 30 September 2014
Accepted 15 April 2015
Available online 11 January 2016

MSC:
primary 54E15, 54D20
secondary 54D35, 54A20

Keywords:
Uniform paracompactness
Strong uniform paracompactness
Cofinal Bourbaki-completeness
Bourbaki-completeness
Bourbaki-boundedness

A B S T R A C T

Recently two new types of completeness in metric spaces, called Bourbaki-completeness and cofinal Bourbaki-completeness, have been introduced in [7]. The purpose of this note is to analyze these completeness properties in the general context of uniform spaces. More precisely, we are interested in how they are related with uniform paracompactness properties, as well as with some kind of uniform boundedness.

1. Introduction

We have recently introduced in [7] two new types of completeness in metric spaces, which we called *Bourbaki-completeness* and *cofinal Bourbaki-completeness*, respectively. They were defined by means of some special classes of Cauchy sequences (equivalently Cauchy nets). The main goal in this note will be to see the behavior of these completeness properties when they are considered in the general frame of uniform spaces and specially how they are related with some kind of uniform paracompactness properties defined in this setting.

In [12], Rice introduced the notion of uniform paracompactness in order to find a convenient formulation of paracompactness for (Hausdorff) uniform spaces. He defined a uniform space (X, μ) to be *uniformly paracompact* if each open cover of X has a uniformly locally finite open refinement. Recall that a cover $\mathcal{A}$ is said to be *uniformly locally finite* if there exists some uniform cover $\mathcal{U} \in \mu$, such that each member of

☆ Partially supported by MINECO Project MTM2012-34341 (Spain).
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http://dx.doi.org/10.1016/j.topol.2015.12.079
0166-8641/

$\mathcal{U}$ meets only finitely many members of $\mathcal{A}$. However, uniform paracompactness was previously introduced by Corson in [4] under a kind of completeness condition involving the so-called *weakly Cauchy* filters. More precisely, Corson proved that a Tychonoff space is paracompact if and only if there exists a compatible uniformity for which every weakly Cauchy filter clusters. It was Smith, the reviewer of Rice's paper, who noticed in [13] that both approaches to uniform paracompactness are in fact equivalent.

On the other hand, in [10] Howes proved a result analogous to the one by Corson, by means of cofinally Cauchy nets. Namely, he proved that a uniform space is uniformly paracompact if and only if every cofinally Cauchy net clusters, i.e., when the space is said to be *cofinally complete*. Note that a net $(x_\lambda)_{\lambda\in\Lambda}$ in a uniform space (X, μ) is *cofinally Cauchy* if for every $\mathcal{U} \in \mu$ there is some $U \in \mathcal{U}$ such that the net is cofinally contained in U, i.e. for every $\lambda \in \Lambda$ there exists $\mu \geq \lambda$ satisfying that $x_\mu \in U$. In this line, Howes also proved that a Tychonoff space X is paracompact if and only if (X, u) is cofinally complete, where u is the finest uniformity on X.

Here, we are interested in the study of the uniform version of another paracompactness property, namely strong paracompactness, together with the search of the corresponding equivalent completeness notion. Recall that a Tychonoff space X is said to be *strongly paracompact* if every open cover of X has a star-finite open refinement, where a cover $\mathcal{A}$ is *star-finite* if every $A \in \mathcal{A}$ meets at most finitely many $A' \in \mathcal{A}$. Clearly every strongly paracompact space is paracompact. However not every paracompact space is strongly paracompact, in order to see that it is enough to consider the well-known metric hedgehog with uncountably many spines (see for instance [7]).

We are going to use the next uniform version of strong paracompactness defined by Hohti in [9].

Definition 1. (Hohti, [9]) A uniform space (X, μ) is *uniformly strongly paracompact* if every open cover X has a uniformly star-finite open refinement, where a cover $\mathcal{A}$ is *uniformly star-finite* if there exists $\mathcal{U} \in \mu$ such that for every $A \in \mathcal{A}$, $St(A, \mathcal{U})$ meets at most finitely many $A' \in \mathcal{A}$.

Note that we are using the following notation. For covers $\mathcal{A}$ and $\mathcal{U}$ of X and for $A \in \mathcal{A}$, we write,

$$St(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$$

and $St(\mathcal{A}, \mathcal{U}) = \{St(A, \mathcal{U}) : A \in \mathcal{A}\}$. Moreover, we will also denote:

$$\begin{gathered} A^1 = A \\ A^2 = St(A, \mathcal{A}),\ \mathcal{A}^2 = \{A^2 : A \in \mathcal{A}\}, \\ \text{for } m \geq 3,\ A^m = St(A^{m-1}, \mathcal{A}),\ \mathcal{A}^m = \{A^m : A \in \mathcal{A}\}, \\ A^\infty = \textstyle\bigcup_{m\in\mathbb{N}} A^m,\ \mathcal{A}^\infty = \{A^\infty : A \in \mathcal{A}\}. \end{gathered}$$

The notion of completeness that we will prove to be equivalent to this uniform strong paracompactness, is the above mentioned *cofinally Bourbaki-completeness*.

Definition 2. ([7]) A net $(x_\lambda)_{\lambda\in\Lambda}$ in a uniform space (X, μ) is *cofinally Bourbaki–Cauchy* if for every $\mathcal{U} \in \mu$ there exist $U \in \mathcal{U}$ and $m \in \mathbb{N}$ such that the net is cofinally contained in U^m. Moreover, if every cofinally Bourbaki–Cauchy net clusters then (X, μ) is said to be *cofinally Bourbaki-complete*.

After proving our general result relating paracompactness and completeness (Theorem 5), the rest of Section 2 will be devoted to study some special uniformities that can be considered in a Tychonoff space X. We refer to the *finest* uniformity u, and also to the uniformities $\mathcal{U}_\delta$ and $\mathcal{U}_{sb}$ introduced by García-Máynez in [5] and [6], respectively. Recall that the uniformity u is given by the collection of all the open normal covers on X, or equivalently, by all the open covers of X with a locally finite cozero refinement. Note that