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A note on the dynamics of cyclically permuted direct product maps

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A B S T R A C T

We obtain some results on the dynamics of maps

$$F(x_1, x_2, \ldots, x_n) = \left(f_{\sigma(1)}(x_{\sigma(1)}), f_{\sigma(2)}(x_{\sigma(2)}), \ldots, f_{\sigma(n)}(x_{\sigma(n)})\right)$$

(we call them cyclically permuted direct product maps), defined from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ into itself, where $X_1, X_2, \ldots, X_n$ are general topological spaces, each map $f_{\sigma(i)} : X_{\sigma(i)} \to X_i$ is continuous, $i = 1, \ldots, n$, and σ is a cyclic permutation of $\{1, 2, \ldots, n\}$, $n \geq 2$. We study the topics of (totally) topological transitivity and (weakly) topological mixing for cyclically permuted direct product maps from the following point of view: we analyze the relationship between the dynamics of F and that of the compositions $f_{\sigma(i)} \circ \ldots \circ f_{\sigma^n(i)}$, $i \in \{1, \ldots, n\}$.

1. Introduction

Let (X, f) be a discrete dynamical system, that is, X is a topological space and f is a continuous map from X to itself. In general, given two topological spaces X, Y, we denote the set of continuous maps from X onto Y by $C(X, Y)$.

In a discrete dynamical system (X, f) the main goal is to obtain information about the evolution in the large of the parts of the topological space X under the iteration of f. Roughly speaking, the study of this

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behaviour using appropriate topological tools receives the name of *Topological Dynamics*. So, let us present some notions appearing in this setting.

First of all, recall that for an $f \in C(X,X)$, its *n-th iterate* is given by $f^n = f \circ f^{n-1}$, $n \in \mathbb{N} := \{1,2,\dots\}$, and we take f^0 as the identity map in X. We say that $x \in X$ is a *periodic point* of *order* $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $0 < i < n$. We write $\mathrm{P}(f)$ to denote the set of periodic points of f. A subset $A \subseteq X$ is called *invariant* if $f(A) \subseteq A$.

We say that f is *topologically transitive* if for any pair U and V of nonempty open sets of X there is a positive integer n depending on U and V such that $f^n(U) \cap V \neq \emptyset$ (if there is $k \in \mathbb{N}$ such that $f^m(U) \cap V \neq \emptyset$ for all $m \geq k$ we say that f is *topologically mixing)*. If f^s is transitive for all $s \in \mathbb{N}$, f is called *totally transitive*. Finally, a map f is *weakly mixing* if $f \times f$ is transitive (in general, if $\phi \in C(X,X)$ and $\varphi \in C(Y,Y)$ the *direct product map* $\phi \times \varphi \in C(X \times Y, X \times Y)$ is defined to be $(\phi \times \varphi)(x,y) = (\phi(x), \varphi(y))$ for $x \in X, y \in Y$). Notice that if f^n is weakly mixing for some $n \in \mathbb{N}$ then f is also weakly mixing.

The relationship between the above notions is the following (see [3]):

$$\text{mixing } \Rightarrow \text{weakly mixing } \Rightarrow \text{totally transitive } \Rightarrow \text{transitive.} \tag{1.1}$$

In particular, in the interval case, $I := [0,1]$, we obtain (see [3])

$$(\text{mixing } \Leftrightarrow \text{weakly mixing } \Leftrightarrow \text{totally transitive}) \Rightarrow \text{transitive.} \tag{1.2}$$

For $n \geq 2$, consider the topological spaces $X_1, X_2, \dots, X_n$ and let $X_\Pi := \prod_{j=1}^n X_j$ denote the Cartesian product, endowed with the well-known product topology. We say that $F : X_\Pi \to X_\Pi$ is a *cyclically permuted direct product* map (for short, c.p.d.p. map) if

$$F(x_1, x_2, \dots, x_n) = \left(f_{\sigma(1)}(x_{\sigma(1)}), f_{\sigma(2)}(x_{\sigma(2)}), \dots, f_{\sigma(n)}(x_{\sigma(n)})\right),$$

where $x_i \in X_i$, $f_{\sigma(i)} : X_{\sigma(i)} \to X_i$ is continuous, $i = 1,\dots,n$, and σ is a cyclic permutation of $\{1,2,\dots,n\}$. The set of c.p.d.p. maps will be denote by $C_A(X_\Pi, X_\Pi)$. The reason for choosing this notation lies in the fact that they also appear in the literature as antitriangular maps (see for instance [1], even they are called σ-permutation maps, see [2]). Notice that F^n is a direct product map, $F^n = \varphi_1 \times \dots \times \varphi_n$, where each $\varphi_j : X_j \to X_j$ is given by

$$\varphi_j = f_{\sigma(j)} \circ f_{\sigma^2(j)} \circ \dots \circ f_{\sigma^n(j)}, \; j = 1,\dots,n. \tag{1.3}$$

When $n = 2$ and $X_1 = X_2 = [0,1]$, this type of maps appears associated with certain economical model so called Cournot duopoly (see [6,10,4], etc.). In this case $F(x_1, x_2) = (R_2(x_2), R_1(x_1))$ and R_1, R_2 are called *reaction maps*. Even we find them in age-structured population models, as in [7], where it is analyzed the Leslie model

$$\begin{cases} x_1(m+1) = x_N(m) g(x_N(m)), \\ x_2(m+1) = x_1(m), \\ \dots \\ x_N(m+1) = x_{N-1}(m), \end{cases}$$

where g is a C^1-map and each variable $x_j(m)$, $j = 1,\dots,N$, $m = 0,1,2,\dots$, determines the population size of the j-age class in the m-th period, being $x_j(0)$ the initial population. To study the behaviour of this model is equivalent to discuss the dynamics of the c.p.d.p. map $F(x_1, x_2, \dots, x_N) = (x_N g(x_N), x_1, \dots, x_{N-1})$, where $\sigma(j) = (j-1) \pmod N$, $j = 1,\dots,N$.